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THE OSCILLATING CIRCULAR AIRFOIL ON THE BASIS OF POTENTIAL THEORY

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THE OSCILLATING CIRCULAR AIRFOIL ON THE BASIS OF POTENTIAL THEORY[1]

PART I

By Th. Schade

Proceeding from the thesis by W. Kinner the present report treats the problem of the circular airfoil in uniform airflow executing small oscillations, the amplitudes of which correspond to whole functions of the second degree in x and y. The pressure distribution is secured by means of Prandtl's acceleration potential. It results in a system of linear equations the coefficients of which can be calculated exactly with the aid of exponential functions and Hankel's functions. The equations necessary are derived in part I; the numerical calculation follows in part II.

INTRODUCTION

The present study was undertaken in connection with Kinner's thesis (reference 1). It deals with the corresponding problem of the oscillating circular airfoil.

The specific quantity necessary for predicting the forces and moments of an oscillating and deforming airfoil is the z coordinate of the lifting surface, $z = z(x, y, t)$. The downwash function $W(x, y, t)$ stands with it in the relationship

$$\frac{\partial z}{\partial t} + V \frac{\partial z}{\partial x} = W$$

Since all oscillations and deformations of the second degree in x and y are computed, the total downwash functions necessary thereto are represented by

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[1] "Theorie der schwingenden kreisförmigen Tragfläche auf potentialtheoretischer Grundlage." Luftfahrtforschung, vol. 17, no. 11-12, Dec. 10, 1940, pp. 387-400, and vol. 19, no. 8, Aug. 20, 1942, pp. 282-291.

$$W = [A + Bx + Cy + Dx^2 + Exy + Fy^2]\, e^{i\nu t}$$

whereby the constants A . . . F may assume any complex values.

For abbreviation of the subsequent formulas the following symbolic style of writing is introduced

$$\frac{e^{+\omega y}}{2}\underbrace{\int_0^\omega dz \int_0^z dz \int_0^z dz \;.\;.\;.\; \int_0^z} f(z\; x_0)\, e^{-zy}\, dz$$

$$\pm\; \frac{e^{-\omega y}}{2}\underbrace{\int_0^\omega dz \int_0^z dz \int_0^z dz \;.\;.\;.\; \int_0^z} f(z\; x_0)\, e^{+zy}\, dz =$$

$$= \underbrace{\int_0^\omega\int\int \;.\;.\;.\; \int} f(z\; x_0) \begin{matrix}\text{Sinh}\\ \text{Cosh}\end{matrix} \Big[[\omega - z] \;\; y \Big]\, dz$$

$$\frac{e^{+\omega i z\cos\alpha}}{(z\cos\alpha + i\sin\alpha)^n} + \frac{e^{+\omega i z\cos\alpha}}{(z\cos\alpha - i\sin\alpha)^n} = 2\,R\left(\frac{e^{\omega i z\cos\alpha}}{(z\cos\alpha + i\sin\alpha)^n}\right)$$

$$\frac{e^{+\omega i z\cos\alpha}}{(z\cos\alpha + i\sin\alpha)^n} - \frac{e^{+\omega i z\cos\alpha}}{(z\cos\alpha - i\sin\alpha)^n} = 2\,I\left(\frac{e^{\omega i z\cos\alpha}}{(z\cos\alpha - i\sin\alpha)^n}\right)$$

THE VELOCITY POTENTIAL (reference 2)

The flow velocity of the circular airfoil is constant (V), the x-axis in direction of V points rearward, the y-axis in spar direction to the right, the z-axis at right angles upward (fig. 1).

With $\underline{w}$ denoting the velocity vector, $\underline{b}$ the acceleration vector and t the time, the relation

$$\underline{b} = \frac{\partial \underline{w}}{\partial t} = V \frac{\partial \underline{w}}{\partial x}$$

is valid.

Then the introduction of $\underline{b} = \operatorname{grad} \varphi$ and $\underline{w} = \operatorname{grad} \Phi$ followed by integration yields

$$\varphi = \frac{\partial \Phi}{\partial t} + V \frac{\partial \Phi}{\partial x}$$

As shown later on, φ indicates, up to a certain factor, the pressure at the upper and lower surface of the disk direct; hence φ may also be termed the pressure potential.

Then if the pressure potential $\varphi(x, y, t)$ is given, the velocity potential Φ can be computed from the above equation:

vel. pot. accel. potential

$$\Phi (x, y, z, t) = \frac{1}{V} \int_{-\infty}^{x} \varphi \left(x', y, z, t - \frac{x - x'}{V} \right) dx'$$

The integration is carried out after putting

$$x' = x - V (t - t')$$

where $x, y, 0,$ and t indicate the coordinates of the starting point. The pressure potential φ satisfies Laplace's equation

$$\Delta\varphi = 0$$

whence, after putting

$$\varphi = C \psi (x, y, z) f(t)$$

ψ (x, y, z) must satisfy the condition $\Delta\psi = 0$. For the present $f(t)$ is any time function and C a constant. The time function follows from the required form of downwash. But since this shall be harmonic in t, $f(t)$ itself must be a harmonic function. Then the vertical component of the velocity is

$$w = \frac{\partial\Phi}{\partial z}$$

The form of the lifting surface follows from

$$w = \frac{\partial z}{\partial t} + V\frac{\partial z}{\partial x}$$

at

$$z = \frac{1}{V}\int_0^x w\left(x', y, 0, t - \frac{x - x'}{V}\right) dx' + F\left(y, t - \frac{x}{V}\right)$$

where function F serves to assume any position of the wing leading edge.

The pressure potential φ to be defined then must be a discontinuous function on the circular surface, but a continuous function at all other points in space.

THE POTENTIAL FUNCTIONS OF THE FIRST KIND

For the sake of brevity the reader is referred to Kinner's report (reference 1) where these problems are treated in detail.

Laplace's equation

$$\Delta\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2} = 0$$

is transformed to elliptic coordinates through the orthogonal transformation

$$x = \sqrt{1 - \mu^2} \quad \sqrt{1 + \eta^2} \cos \vartheta$$

$$y = \sqrt{1 - \mu^2} \quad \sqrt{1 + \eta^2} \sin \vartheta$$

$$z = \mu \, \eta$$

The x, y, z coordinates are made dimensionless with the base circle radius c.

With the application of a separation theorem, the real potential functions of the first kind with discontinuity at the disk then are according to Kinner (reference 1):

$$\psi_n^m (\mu, \eta, \vartheta) = \frac{1}{i^{n - m + 1}} \frac{(n - m)!}{(n + m)!} C_n^m P_n^m (\mu) Q_n^m (i\eta) \begin{Bmatrix} \cos m \vartheta \\ \sin m \vartheta \end{Bmatrix}$$

$n + m$ being posted odd, C_n^m denoting a constant factor, and factor $\frac{(n - m)!}{(n + m)!}$ merely serving to simplify the calculation.

DOWNWASH

The downwash is then computed for the potential functions previously enumerated, resulting in the following downwash expressions:

$$W = f(x, y) \, e^{i\upsilon t} + e^{i(\upsilon t - \omega x)} \, g(y) \tag{3.1}$$

where $f(x, y)$ is a whole rational function, but $e^{-i\omega x} g(y)$ is transcendental in the second term.

The downwash functions are so selected and linearly superimposed that $f(x, y)$ exactly represents the required downwash.

The term containing the factor $e^{i(\upsilon t - \omega x)}$ is then inadequate still for representing the disk motion. Subsequently, it is linearly superposed with the downwash functions of the second type, so that this total term disappears. The solution of this problem is hereinafter termed "compensation calculus," a detailed discussion of which is given in the section, Compensation Calculus. It affords the linear equation systems, the solution of which gives the coefficients for the prediction of the pressure distribution.

To compute the downwash, it is necessary to form

$$W_n^m = \frac{1}{V}\int_{-\infty}^{x}\left(\frac{\varphi_n^m}{z}\right)_{z \to 0} dx \tag{3.2}$$

the integration taking place on a streamline.

With $x_o = \pm\sqrt{1 - y^2}$ denoting the x coordinate of the disk edge and $\frac{r}{c} = \sqrt{x^2 + y^2 + z^2}$ the radius of any sphere about the disk center, the total integration space can be divided in the outer space with $r > c$ and the inner space with $r < c$.

It proved advantageous to carry out the integration with respect to x according to (3.2) separately for the outer and inner space.

Equation (3.2) is written:

$$W_n^m = \frac{1}{V}\left[\int_{-\infty}^{-x_o}\left(\frac{\varphi_n^m}{z}\right)_{z \to 0} dx + \int_{-x_o}^{x}\left(\frac{\varphi_n^m}{z}\right)_{z \to 0} dx\right] = W_{n,a}^m + W_{n,i}^m$$

For the integration over x in the plane of the disk $(z = 0)$ the following formulas should be considered:

1. For the outer space:

$$x \lessgtr 0 \qquad \mu = 0 \qquad x_o^2 = 1 - y^2$$

$$dx = \mp\frac{\eta\, d\eta}{\sqrt{x_o^2 + \eta^2}} \qquad \cos\vartheta = \mp\sqrt{\frac{x_o^2 + \eta^2}{1 + \eta^2}}$$

$$\frac{d\mu}{dz} = \frac{1}{\eta} \qquad \frac{d\eta}{dz} = 0 \qquad \frac{d\vartheta}{dz} = 0$$

2. For the inner space:

$$x \lessgtr 0 \qquad \eta = 0$$

$$dx = \mp \frac{\mu \, d\mu}{\sqrt{x_o^2 - \mu^2}} \qquad \cos\vartheta = \mp \sqrt{\frac{x_o^2 - \mu^2}{1 - \mu^2}}$$

$$\frac{\partial\mu}{\partial z} = 0 \qquad \frac{\partial\eta}{\partial z} = \frac{1}{\mu} \qquad \frac{\partial\vartheta}{\partial z} = 0$$

REPRESENTATION OF FLAPPING MOTION

The requisite downwash formula for the flapping motion reads

$$W = A \, e^{i\nu t} \tag{4.1}$$

the pressure potential

$$\varphi_1^o = C_1^o \, \mu(1 - \eta \operatorname{arc\,ctg} \eta) e^{i\nu t} \tag{4.2}$$

being used for the representation.

Then the downwash follows at

$$W_1^o = \frac{C_1^o}{V} \int_{-\infty}^{x} \lim_{z \to o} \frac{\partial \varphi_1^o}{\partial z} \, dx'$$

which gives for inner space

$$W_{1,i}^o = \frac{C_1^o}{V} \int_{-x_o}^{x} -\frac{\pi}{2} e^{i\nu t'} \, dx'$$

and with $\nu(t-t') = \omega(x-x')$; from $\omega = \frac{\nu c}{V}$ follows[1]

$$W^0_{1,i} = -\frac{\pi}{2}\frac{C^0_1}{V} e^{i(\nu t-\omega x)} \int_{-x_0}^{x} e^{i\omega x'}\, dx'$$

$$W^0_{1,i} = -\frac{\pi}{2}\frac{C^0_1}{V}\cdot\frac{1}{\omega i} e^{i\nu t} + \frac{\pi}{2}\cdot\frac{C^0_1}{V}\cdot\frac{e^{i(\nu t-\omega x)}}{\omega i} e^{-\omega i x_0}.$$

The downwash for the outer space is:

$$W^0_{1,a} = \frac{C^0_1}{V} e^{i(\nu t-\omega x)} \int_0^\infty (1-\eta\,\mathrm{arc\,ctg}\,\eta)\, \frac{e^{-\omega i\sqrt{x_0^2+\eta^2}}}{\sqrt{x_0^2+\eta^2}}\, d\eta.$$

For the solution of the integral the following substitution is made, and this is equally applied on all further outer space integrals:

$$\eta = x_0\sqrt{s^2-1} \qquad \sqrt{x_0^2+\eta^2} = x_0 s$$

$$\frac{d\eta}{\sqrt{x_0^2+\eta^2}} = \frac{ds}{\sqrt{s^2-1}}.$$

In the subsequent calculations the factor $\frac{C^0_1}{V} e^{i(\nu t-\omega x)}$ is then omitted, and indicated by dash over W.

Herewith:

$$\overline{W}^0_{1,a} = \int_1^\infty \left(1 - x_0\sqrt{s^2-1}\,\mathrm{arc\,ctg}\, x_0\sqrt{s^2-1}\right) \frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\, ds \quad . \; . \; . \; (4,3)$$

and, after partial integration:

$$\overline{W}^0_{1,a} = \frac{1}{\omega i x_0}\left[x_0\cdot\mathrm{arc\,ctg}\, x_0\sqrt{s^2-1}\cdot e^{-\omega i x_0 s}\right]_1^\infty + \int_1^\infty \frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\, ds + \frac{1}{\omega i}\int_1^\infty \frac{x_0 s}{x_0^2 s^2+1-x_0^2}\cdot\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\, ds.$$

whence the downwash of the outer space

$$\overline{W}^0_{1,a} = -\frac{\pi}{2\omega i} e^{-\omega i x_0} - \frac{\pi}{2} i H_0^{(2)}(\omega\cdot x_0) + \frac{1}{\omega i}\int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}}, \quad . \; . \; . \; . \; . \; (4,4)$$

where $-\frac{\pi}{2} i H_0^{(2)}(\omega\cdot x_0) = \int_1^\infty \frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\, ds$

denotes Hankel's cylindrical function. The solution of the determinate integral over s is achieved by transformation in an indeterminate integral over ω. Putting $y = \sin\alpha$, $x_0 = \cos\alpha$, then yields

$$\int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} = \frac{1}{2}\int_1^\infty \frac{e^{-\omega i\cos\alpha\cdot s}}{\sqrt{s^2-1}}\left(\frac{1}{s\cdot\cos\alpha+i\sin\alpha} + \frac{1}{s\cdot\cos\alpha - i\sin\alpha}\right) ds =$$

$$= \frac{1}{2}\left[e^{-\omega\sin\alpha}\int_1^\infty \frac{e^{-\omega i(s\cos\alpha+i\sin\alpha)}}{(s\cdot\cos\alpha+i\sin\alpha)\sqrt{s^2-1}}\, ds + e^{\omega\sin\alpha}\int_1^\infty \frac{e^{-\omega i(s\cdot\cos\alpha-i\sin\alpha)}}{(s\cdot\cos\alpha-i\sin\alpha)\sqrt{s^2-1}}\, ds\right].$$

In this case, however, the relations

$$\frac{d}{d\omega}\int_1^\infty \frac{e^{-\omega i(s\cos\alpha+i\sin\alpha)}}{(s\cdot\cos\alpha+i\sin\alpha)\sqrt{s^2-1}}\, ds = -i\int_1^\infty \frac{e^{-\omega i(s\cos\alpha+i\sin\alpha)}}{\sqrt{s^2-1}}\, ds = -\frac{\pi}{2}\cdot e^{\omega\sin\alpha} H_0^{(2)}(\omega x_0),$$

$$\frac{d}{d\omega}\int_1^\infty \frac{e^{-\omega i(s\cos\alpha-i\sin\alpha)}}{(s\cdot\cos\alpha-i\sin\alpha)\sqrt{s^2-1}}\, ds = -i\int_1^\infty \frac{e^{-\omega i(s\cdot\cos\alpha-i\sin\alpha)}}{\sqrt{s^2-1}}\, ds = -\frac{\pi}{2}\, e^{-\omega\sin\alpha} H_0^{(2)}(\omega x_0),$$

hold true, and with it the above integral becomes:

$$\int_1^\infty \frac{x_0\cdot s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} =$$

$$= \frac{1}{2} e^{-\omega\sin\alpha}\left[C_1^{(1)} - \int_0^\omega \frac{\pi}{2} H_0^{(2)}(\overline{\omega}\cdot\cos\alpha)\cdot e^{+\overline{\omega}\sin\alpha}\, d\overline{\omega}\right] + \frac{1}{2} e^{+\omega\sin\alpha}\left[C_2^{(1)} - \int_0^\omega \frac{\pi}{2} H_0^{(2)}(\overline{\omega}\cos\alpha)\cdot e^{-\overline{\omega}\sin\alpha}\, d\overline{\omega}\right] =$$

$$= \frac{1}{2} C_1^{(1)}\cdot e^{-\omega\sin\alpha} + \frac{1}{2} C_2^{(1)}\cdot e^{+\omega\sin\alpha} - \frac{\pi}{2}\int_0^\omega H_0^{(2)}(\overline{\omega}\cdot\cos\alpha)\cdot\mathfrak{Cof}\,[(\omega-\overline{\omega})\sin\alpha]\, d\overline{\omega} =$$

$$= i Q_0(y)\,\mathfrak{Sin}\,(\omega\sin\alpha) + \frac{\pi}{2} P_0(y)\cdot\mathfrak{Cof}\,(\omega\sin\alpha) - \frac{\pi}{2}\int_0^\omega H_0^{(2)}(\overline{\omega}\cos\alpha)\cdot\mathfrak{Cof}\,[(\omega-\overline{\omega})\sin\alpha]\, d\overline{\omega}, \quad . \; . \; . \; . \; . \; . \; (4,5)$$

the constants of integration $C_1^{(1)}$ and $C_2^{(1)}$ being computed according to (reference 3):

$$\left.\begin{aligned} C_{\frac{1}{2}}^{(2K+1)} &= \int_1^\infty \frac{1}{(s\cos\alpha\pm i\sin\alpha)^{2K+1}}\frac{ds}{\sqrt{s^2-1}} = (-1)^K\left(\frac{\pi}{2} P_{2K} \mp i Q_{2K}\right) \\ C_{\frac{1}{2}}^{(2K)} &= \int_1^\infty \frac{1}{(s\cos\alpha\pm i\sin\alpha)^{2K}}\frac{ds}{\sqrt{s^2-1}} = (-1)^K\left(Q_{2K-1} \pm i\frac{\pi}{2} P_{2K-1}\right) \end{aligned}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; (4,6)$$

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[1] ω here denotes the real reduced frequency, while $\overline{\omega}$ in the subsequent calculations are integration variables.

The total downwash is

$$W_1^0 = W_{1,i}^0 + W_{1,a}^0$$

and herewith

$$W_1^0 = -\frac{\pi}{2}\cdot\frac{C_1^0}{V}\cdot\frac{e^{i\nu t}}{\omega i} + \frac{C_1^0}{V}\cdot e^{i(\nu t-\omega x)}\Big[-\frac{\pi}{2} i H_0^{(2)}(\omega\cdot\cos\alpha) +$$

$$+\frac{\pi i}{2\omega}\int_0^{\omega} H_0^{(2)}(\overline{\omega}\cos\alpha)\cdot \mathfrak{Cof}\,[(\omega-\overline{\omega})\sin\alpha]\,d\overline{\omega} +$$

$$+\frac{1}{2\omega}\mathfrak{Sin}\,(\omega\sin\alpha)\cdot\ln\frac{1+\sin\alpha}{1-\sin\alpha} +$$

$$+\frac{\pi}{2\omega i}\cdot\mathfrak{Cof}\,(\omega\cdot\sin\alpha)\Big] \quad \ldots\ldots\ldots\ldots \quad (4,7)$$

The indeterminate integral over ω is definitely computable and is given later as series in ω.

REPRESENTATION OF TORSIONAL OSCILLATION ABOUT THE Y-AXIS

(RELAXATION OSCILLATION)

The requisite downwash formula shall read:

$$W = B\cdot x\cdot e^{i\nu t} \quad \ldots\ldots\ldots \quad (5,1)$$

This furnishes the potential function

$$\varphi_2^1 = \frac{C_2^1}{V}\cdot\frac{\mu}{2}\sqrt{1-\mu^2}\sqrt{1+\eta^2}\times$$

$$\times\left(3-\frac{1}{1+\eta^2}-3\eta\,\mathrm{arc\,ctg}\,\eta\right)\cos\vartheta\cdot e^{i\nu t} \quad . \; . \quad (5,2)$$

With the given transformations the downwash for the inner space problem reads:

$$W_{2,i}^1 = -\frac{C_2^1}{V}\frac{3}{4}\pi\int_{-x_0}^{x} x'\cdot e^{i\omega x'}\cdot e^{i(\nu t-\omega x)}\,dx' =$$

$$= -\frac{3}{4}\pi e^{i\nu t}\cdot\frac{C_2^1}{V}\left(\frac{1}{\omega i}x+\frac{1}{\omega^2}\right)+$$

$$+\frac{C_2^1}{V}\frac{3}{4}\pi\cdot e^{i(\nu t-\omega x)}\left(-\frac{x_0}{i\omega}+\frac{1}{\omega^2}\right)e^{-\omega i x_0}.$$

and that for the outer space:

$$W_{2,a}^1 = -\frac{C_2^1}{V}\cdot\frac{1}{2}\cdot\int_0^{\infty}\left(3-\frac{1}{1+\eta^2}-3\eta\,\mathrm{arc\,ctg}\,\eta\right)e^{i\nu t'}\,d\eta$$

and with the substitutions given in the preceding section:

$$\overline{W}_{2,a}^1 = -\frac{1}{2}\int_1^{\infty}\Big(3-3x_0\sqrt{s^2-1}\,\mathrm{arc\,ctg}\,x_0\sqrt{s^2-1}-$$

$$-\frac{1}{x_0^2 s^2+1-x_0^2}\Big)\frac{e^{-\omega i x_0 s}\cdot x_0\cdot s}{\sqrt{s^2-1}}\,ds =$$

$$=\frac{3}{2i}\frac{d}{d\omega}\int_1^{\infty}\left(1-x_0\sqrt{s^2-1}\,\mathrm{arc\,ctg}\,x_0\sqrt{s^2-1}\right)\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\,ds+$$

$$+\frac{1}{2}\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}}.$$

By comparing this result with the outer space integral of the flapping oscillation, it can also be written in the form

$$\overline{W}_{2,a}^1 = \frac{3}{2i}\frac{d}{d\omega}\overline{W}_{1,a}^0 + \frac{1}{2}\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}}.$$

which means:

$$W_2^1 = -\frac{C_2^1}{V}\cdot\frac{3\pi}{4}e^{i\nu t}\left(\frac{x}{\omega i}+\frac{1}{\omega^2}\right)+$$

$$+\frac{C_2^1}{V}e^{i(\nu t-\omega x)}\Big[-\frac{3i}{2}\frac{d}{d\omega}\left(-\frac{\pi}{2}iH_0^{(2)}\right)+$$

$$+\left(\frac{1}{2}+\frac{3}{2\omega^2}-\frac{3}{2\omega}\frac{d}{d\omega}\right)\int_1^{\infty}\frac{e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\frac{ds}{\sqrt{s^2-1}}\Big] \quad (5,3)$$

The first term gives the requisite downwash and, in addition, a flapping oscillation, which can be compensated again by linear superposition with $W_1{}^0$. Putting for this purpose $C_2^1 = \omega i K_2^1$

and forming $W_2^1 + \frac{3}{2}W_1^0\frac{K_2^1}{C_1^0} = W_K$.

gives the downwash

$$W_K = -\frac{K_2^1}{V}\frac{3\pi}{4}e^{i\nu t}\cdot x + \frac{K_2^1}{V}e^{i(\nu t-\omega x)}\Big[-\frac{3}{4}\pi i H_0^{(2)}(\omega\cdot x_0) -$$

$$-\frac{3}{4}\pi i\omega\frac{d}{d\omega}H_0^{(2)}(\omega x_0) - \frac{\omega}{2i}\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}\,ds}{(x_0^2 s^2+1-x_0^2)\sqrt{s^2-1}} +$$

$$+\frac{3}{2i}\frac{d}{d\omega}\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{(x_0^2 s^2+1-x_0^2)\sqrt{s^2-1}}\,ds\Big] \quad \ldots \quad (5,4)$$

In this formula the integrals are already known from the calculation of $W_1{}^0$. (See (4.5).) Moreover, this representation is beneficial for the compensation process later on.

REPRESENTATION OF FLEXURAL OSCILLATIONS

In the course of the symmetrical oscillations of the disk the oscillations with parabolic deformation

$W = D\,x^2 e^{i\upsilon t}$ and $W = F\,y^2 e^{i\upsilon t}$, respectively, are dealt with. To provide simpler expressions for the downwash calculation, the following linear combinations are instead used as basis:

$$\left.\begin{aligned} W_{B,1} &= D_1\cdot(x^2+y^2)\cdot e^{i\nu t}\\ W_{B,2} &= F_1\cdot(x^2-y^2)\cdot e^{i\nu t}\end{aligned}\right\} \quad \ldots \quad (6,1)$$

The individual deformations are obtained by additive superposition.

The representation of the first requisite downwash under (6.1) proceeds from the pressure potential

$$\varphi_3^0 = C_3^0 \frac{5\mu^3 - 3\mu}{2}\left(\frac{5}{2}\eta^2 + \frac{2}{3} - \frac{5\eta^3 + 3\eta}{2}\operatorname{arc\,ctg}\eta\right)\cdot e^{i\nu t}$$

The inner space problem yields the downwash

$$W_{3,i}^0 = -\frac{3}{8}\pi\frac{C_3^0}{V}\int_{-x_0}^{x}\left[2 - 5\left(x'^2 + y^2\right)\right]e^{i\nu t'}\,dx' =$$

$$= \frac{C_3^0}{V}\left\{e^{i\nu t}\left(-\frac{3\pi}{4i\omega} + \frac{15\pi}{8i\omega}y^2 - \frac{15\pi}{8i\omega^3}\left(-\omega^2x^2 - 2i\omega x + 2\right)\right) - \right.$$

$$\left. - e^{i(\nu t - \omega x)}\cdot e^{-i\omega x_0}\left[-\frac{3\pi}{4\omega i} + \frac{15\pi}{8i\omega}y^2 - \frac{15\pi}{8i\omega^3}\left(-\omega^2x_0^2 + 2i\omega x_0 + 2\right)\right]\right\}.$$

and the outer space problem:

$$\overline{W}_{3,a}^0 = -\int_0^\infty \frac{3}{2}\left(\frac{5}{2}\eta^2 + \frac{2}{3} - \frac{5\eta^3 + 3\eta}{2}\operatorname{arc\,ctg}\eta\right)\cdot\frac{e^{-\omega i\sqrt{x_0^2 + \eta^2}}}{\sqrt{x_0^2 + \eta^2}}\,d\eta =$$

$$= -\int_1^\infty \frac{3}{2}\left[\frac{5}{2}x_0^2\left(s^2 - 1\right) + \frac{2}{3} - \frac{x_0\sqrt{s^2 - 1}}{2}\left(5x_0^2\left(s^2 - 1\right) + 3\right)\operatorname{arc\,ctg}x_0\sqrt{s^2 - 1}\right]\frac{e^{-\omega i x_0 s}}{\sqrt{s^2 - 1}}\,ds.$$

Then:

$$x_0^2\int_1^\infty\left(s^2 - 1\right)\left(1 - x_0\sqrt{s^2 - 1}\operatorname{arc\,ctg}x_0\sqrt{s^2 - 1}\right)\frac{e^{-\omega i x_0 s}}{\sqrt{s^2 - 1}}\,ds = -\left(\frac{d^2\,\overline{W}_{1,a}^0}{d\omega^2} + x_0^2\cdot\overline{W}_{1,a}^0\right) \quad \ldots\ldots (6,2)$$

That is:

$$\overline{W}_{3,a}^0 = \frac{15}{4}\left(\frac{d^2\,\overline{W}_{1,a}^0}{d\omega^2} + x_0^2\,\overline{W}_{1,a}^0\right) - \frac{9}{4}\overline{W}_{1,a}^0 + \frac{5}{4}\left(-\frac{\pi}{2}\,i\,H_0^{(2)}(\omega x)\right).$$

Insertion of the value according to (4.4) for $\overline{W}_{1,a}^0$ affords

$$W_3^0 = W_{3,i}^0 + W_{3,a}^0 = \frac{C_3^0}{V}e^{i\nu t}\cdot\frac{\pi}{4}\left[-\frac{3}{\omega i} + \frac{15}{2i\omega}y^2 + \frac{15}{2i\omega}\left(x^2 + \frac{2ix}{\omega} - \frac{2}{\omega^2}\right)\right] +$$

$$+ \frac{C_3^0}{V}e^{i(\nu t - \omega x)}\left\{\left(-1 + \frac{15x_0^2}{4} + \frac{15}{4}\frac{d^2}{d\omega^2}\right)\left[-\frac{\pi}{2}\,i\,H_0^{(2)}(\omega x_0)\right] + \right.$$

$$\left. + \left(-\frac{9}{4\omega i} + \frac{15x_0^2}{4\omega i} + \frac{15}{2i\omega^3} - \frac{15}{2i\omega^2}\frac{d}{d\omega} + \frac{15}{4i\omega}\frac{d^2}{d\omega^2}\right)\int_1^\infty\frac{x_0 s\, e^{-\omega i x_0 s}}{x_0^2s^2 + 1 - x_0^2}\,\frac{ds}{\sqrt{s^2 - 1}}\right\} \quad \ldots (6,3)$$

The first part of (6.3) yields the required expression for the downwash of the bending oscillation, combined with a torsional and flapping oscillation, which shall now be compensated.

By putting

$$C_3^0 = \omega i K_3^0$$

and forming

$$W_3^0 - 5W_2^1\cdot\frac{K_3^0}{C_2^1} - \frac{3}{2}W_1^0\cdot\frac{K_3^0}{C_1^0}\cdot\omega i = W_B^1.$$

the utilization of the relation

$$\frac{x_0^2}{\omega i}\int_1^\infty\frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2s^2 + 1 - x_0^2}\cdot\frac{ds}{\sqrt{s^2 - 1}} = \frac{1}{\omega}\,\frac{\pi}{2}\,i\,\frac{d}{d\omega}H_0^{(2)} +$$

$$+ \left(\frac{1}{\omega i} - \frac{1}{\omega i}\frac{d^2}{d\omega^2}\right)\int_1^\infty\frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2s^2 + 1 - x_0^2}\cdot\frac{ds}{\sqrt{s^2 - 1}} \quad (6,4)$$

and of Bessel's differential equation

$$\frac{d^2}{d\omega^2}H_0^{(2)}(\omega\cdot x_0) + x_0^2\cdot H_0^{(2)}(\omega\cdot x_0) = -\frac{1}{\omega}\frac{d}{d\omega}H_0^{(2)}(\omega\cdot x_0) \quad \ldots (6,5)$$

changes (6.3) to

$$W_B^1 = \frac{K_3^0}{V}\cdot\frac{15}{8}\pi\left(x^2 + y^2\right)\cdot e^{i\nu t} +$$

$$+ \frac{K_3^0}{V}\cdot e^{i(\nu t - \omega x)}\left(-\frac{\pi}{2}\cdot\frac{5}{2}\,\omega H_0^{(2)}(\omega\cdot x_0) - \right.$$

$$\left. - \frac{5}{2}\int_1^\infty\frac{x_0 s\cdot e^{-i\omega x_0 s}}{\left(x_0^2s^2 + 1 - x_0^2\right)}\cdot\frac{ds}{\sqrt{s^2 - 1}}\right) \quad \ldots\ldots (6,6)$$

Formula (6.6) gives the requisite downwash and an additive term to be compensated later. The second oscillation given under (6.1) is obtained through the pressure potential

$$\varphi_3^2 = C_3^2 \frac{-\mu^3+\mu}{8}(1+\eta^2)\left(15-\frac{5}{1+\eta^2}-\frac{2}{(1+\eta^2)^2}-\right.$$
$$\left.-15\,\eta \operatorname{arc\,ctg}\eta\right)\cos 2\vartheta \cdot e^{i\nu t} \quad \ldots\ldots\ldots\ldots \quad (6,7)$$

The downwash of the inner space then reads

$$W_{3,i}^2 = \frac{C_3^2}{V}\int_{-x_0}^{x} -\frac{15}{16}\pi\,(x'^2-y^2)\,e^{i\nu t'}\,dx'$$

$$W_{3,i}^2 = \frac{C_3^2}{V}\left[\frac{15}{16}\pi\cdot\frac{1}{i\omega^3}(-\omega^2x^2-2i\omega x+2)+\frac{15}{16\,i\omega}\pi y^2\right]e^{i\nu t}+$$
$$+\frac{C_3^2}{V}e^{i(\nu t-\omega x)}\left[\frac{15}{16}\pi\cdot\frac{1}{i\omega^3}(+\omega^2x_0^2-2i\omega x_0-2)-\right.$$
$$\left.-\frac{15}{16\,i\omega}\pi y^2\right]e^{-i\omega x_0} \quad \ldots\ldots\ldots\ldots \quad (6,8)$$

and for the outer space

$$W_{3,a}^2 = \frac{C_3^2}{V}\int_0^\infty \frac{1}{8}(1+\eta^2)\left(15-\frac{5}{1+\eta^2}-\frac{2}{(1+\eta^2)^2}-\right.$$
$$\left.-15\,\eta\operatorname{arc\,ctg}\eta\right)\frac{\cos 2\vartheta}{\sqrt{x_0^2+\eta^2}}e^{i\nu t'}\,d\eta,$$

with

$$\cos\vartheta = -\frac{\sqrt{x_0^2+\eta^2}}{\sqrt{1+\eta^2}} \qquad \cos 2\vartheta = \frac{2(x_0^2+\eta^2)}{1+\eta^2}-1.$$

or, after substitution in s:

$$\overline{W}_{3,a}^2 = \int_1^\infty \frac{1}{8}(15-15x_0\sqrt{s^2-1}\operatorname{arc\,ctg}x_0\sqrt{s^2-1})\frac{x_0^2s^2\cdot e^{-\omega i x_0 s}}{\sqrt{s^2-1}}ds-\frac{1}{8}(1-x_0^2)\int_1^\infty(15-15x_0\sqrt{s^2-1}\operatorname{arc\,ctg}x_0\sqrt{s^2-1})\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}ds-$$
$$-\frac{1}{8}\int_1^\infty\left(\frac{5}{s^2x_0^2+1-x_0^2}+\frac{2}{(x_0^2s^2+1-x_0^2)^2}\right)(x_0^2s^2-1+x_0^2)\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}ds.$$

With $W_{1,a}^0$ and its derivatives, the total downwash follows (cf. (6.2)) at:

$$W_3^2 = W_{3,i}^2 + W_{3,a}^2$$

$$W_3^2 = \frac{C_3^2}{V}e^{i\nu t}\cdot\frac{15\pi}{16}\left(-\frac{1}{i\omega}x^2-\frac{2}{\omega^2}x+\frac{2}{i\omega^3}+\frac{1}{i\omega}y^2\right)+\frac{C_3^2}{V}e^{i(\nu t-\omega x)}\left[\left(-\frac{15}{8}(1-x_0^2)-\frac{15}{8}\frac{d^2}{d\omega^2}\right)\left(-\frac{\pi}{2}iH_0^{(2)}(\omega x_0)\right)+\right.$$
$$+\left(\frac{-15}{4i\omega^2}-\frac{15}{8}\frac{(1-x_0^2)}{\omega i}+\frac{15}{4i\omega^2}\frac{d}{d\omega}-\frac{15}{8i\omega}\frac{d^2}{d\omega^2}\right)\int_1^\infty\frac{x_0 s\, e^{-\omega i x_0 s}}{x_0^2s^2+1-x_0^2}\frac{ds}{\sqrt{s^2-1}}-$$
$$\left.-\frac{1}{8}\int_1^\infty\left(\frac{5}{x_0^2s^2+1-x_0^2}+\frac{2}{(x_0^2s^2+1-x_0^2)^2}\right)(x_0^2s^2-1+x_0^2)\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}ds\right].$$

With (6.4) and (6.5) and the relation

$$\int_1^\infty\frac{x_0^2s^2-1+x_0^2}{x_0^2s^2+1-x_0^2}\cdot\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}ds = -\int_1^\infty\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}ds-\frac{2}{i}\frac{d}{d\omega}\int_1^\infty\frac{x_0 s\cdot e^{-\omega i x_0 s}}{(x_0^2s^2+1-x_0^2)\sqrt{s^2-1}}ds,$$

it then affords

$$W_3^2 = \frac{C_3^2}{V}\frac{15\pi}{16}e^{i\nu t}\left(-\frac{1}{i\omega}x^2+\frac{1}{2\omega}y^2-\frac{2}{\omega^2}x+\frac{2}{i\omega^3}\right)+\frac{C_3^2}{V}e^{i(\nu t-\omega x)}\left[\left(-\frac{5}{4}-\frac{15}{4\omega}\frac{d}{d\omega}-\frac{15}{4}\frac{d^2}{d\omega^2}\right)\left(-\frac{\pi}{2}iH_0^{(2)}(\omega x_0)\right)+\right.$$
$$\left.+\left(-\frac{15}{8\omega i}-\frac{15}{4i\omega^2}+\frac{5}{4i}\frac{d}{d\omega}+\frac{15}{4i\omega^2}\frac{d}{d\omega}-\frac{15}{8i\omega}\frac{d^2}{d\omega^2}\right)\int_1^\infty\frac{e^{-\omega i x_0 s}}{x_0^2s^2+1-x_0^2}\frac{ds}{\sqrt{s^2-1}}-\frac{1}{4}\int_1^\infty\frac{x_0^2s^2-1+x_0^2}{(x_0^2s^2+1-x_0^2)^2}\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}ds.\right.$$
$$\ldots \quad (6,9)$$

Analogously

$$\int_1^\infty\frac{x_0^2s^2-1+x_0^2}{(x_0^2s^2+1-x_0^2)^2}\cdot\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}ds =$$
$$=\frac{1}{2}\left[e^{-\omega\sqrt{1-x_0^2}}\int_1^\infty\frac{e^{-\omega i(x_0 s+i\sqrt{1-x_0^2})}}{(x_0 s+i\sqrt{1-x_0^2})^2\sqrt{s^2-1}}ds+e^{+\omega\sqrt{1-x_0^2}}\int_1^\infty\frac{e^{-\omega i(x_0 s-i\sqrt{1-x_0^2})}}{(x_0 s-i\sqrt{1-x_0^2})^2\sqrt{s^2-1}}ds\right].$$

For $y=\sin\alpha$; $x_0=\cos\alpha$ with allowance for

$$\frac{d}{d\omega}\int_1^\infty\frac{e^{-\omega i(s\cdot\cos\alpha+i\sin\alpha)}}{(s\cdot\cos\alpha+i\sin\alpha)^2\sqrt{s^2-1}}ds = -i\int_1^\infty\frac{e^{-\omega i(s\cdot\cos\alpha+i\sin\alpha)}}{(s\cdot\cos\alpha+i\sin\alpha)\sqrt{s^2-1}}ds,$$

$$\frac{d}{d\omega}\int_1^\infty\frac{e^{-\omega i(s\cdot\cos\alpha-i\sin\alpha)}}{(s\cdot\cos\alpha-i\sin\alpha)^2\sqrt{s^2-1}}ds = -i\int_1^\infty\frac{e^{-\omega i(s\cdot\cos\alpha-i\sin\alpha)}}{(s\cdot\cos\alpha-i\sin\alpha)\sqrt{s^2-1}}ds,$$

it is

$$\int_1^{\infty}\frac{x_0^2 s^2-1+x_0^2}{(x_0^2 s^2+1-x_0^2)^2}\cdot\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\,ds=$$

$$=\frac{1}{2}\left[e^{-\omega\sin\alpha}\left(C_1^{(2)}-i\int_1^{\infty}\int_0^{\omega}\frac{e^{-\overline{\omega} i(s\cdot\cos\alpha+i\sin\alpha)}}{(s\cdot\cos\alpha+i\sin\alpha)\sqrt{s^2-1}}\,ds\,d\overline{\omega}\right)+e^{+\omega\sin\alpha}\left(C_2^{(2)}-i\int_1^{\infty}\int_0^{\omega}\frac{e^{-\overline{\omega} i(s\cdot\cos\alpha-i\sin\alpha)}}{(s\cdot\cos\alpha-i\sin\alpha)\sqrt{s^2-1}}\,ds\,d\overline{\omega}\right)\right]$$

and, by similar considerations

$$\int_1^{\infty}\frac{x_0^2 s^2-1+x_0^2}{(x_0^2 s^2+1-x_0^2)^2}\cdot\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\,ds=\frac{1}{2}\left[e^{-\omega\sin\alpha}\left\{C_1^{(2)}-i\,C_1^{(1)}\cdot\omega-\int_1^{\infty}\int_0^{\omega}\int_{*}\frac{e^{-\overline{\omega} i(s\cdot\cos\alpha+i\sin\alpha)}}{\sqrt{s^2-1}}\,d\overline{\omega}\,ds^{3})\right\}+\right.$$

$$\left.+e^{\omega\sin\alpha}\left\{C_2^{(2)}-i\,C_2^{(1)}\cdot\omega-\int_1^{\infty}\int_0^{\omega}\int_{*}\frac{e^{-\overline{\omega} i(\cos\alpha\cdot s-i\sin\alpha)}}{\sqrt{s^2-1}}\,ds\,d\overline{\omega}\right\}\right]\quad\ldots\ldots\quad(6,10)$$

Constants $C^{(2)}_{\substack{1\\2}}$ and $C^{(1)}_{\substack{1\\2}}$ are again computed by changing to the stationary value $\omega\longrightarrow 0$ according to (4.6).

These quantities posted in (6.10) followed by the integration in the triple integrals over s, which gives Hankel's function, result in

$$\int_1^{\infty}\frac{x_0^2 s^2-1+x_0^2}{(x_0^2 s^2+1-x_0^2)^2}\,\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\,ds=-Q_1(y)\cdot\mathfrak{Cof}\,(\omega\sin\alpha)+i\frac{\pi}{2}P_1(y)\cdot\mathfrak{Sin}\,(\omega\sin\alpha)-$$

$$-i\frac{\pi}{2}\cdot\omega P_0(y)\cdot\mathfrak{Cof}\,(\omega\sin\alpha)+\omega\cdot Q_0(y)\cdot\mathfrak{Sin}\,(\omega\sin\alpha)+\frac{\pi}{2}i\int_0^{\omega}\int_{*}H_0^{(2)}(\overline{\omega}\cdot\cos\alpha)\cdot\mathfrak{Cof}\,[(\omega-\overline{\omega})\sin\alpha]\,d\overline{\omega}.$$

Putting

$$C_3^2=i\,\omega\,K_3^2$$

and compensating the torsional oscillation still existing in (6.9) by forming

$$W_B^2=W_3^2+\frac{5}{2}\cdot\frac{K_3^2}{C_2^1}W_-^1.$$

the downwash formula (6.9) then becomes

$$W_B^2=\frac{K_3^2}{V}\cdot\frac{15}{16}\pi(y^2-x^2)e^{i\nu t}+\frac{K_3^2}{V}e^{i(\nu t-\omega x)}\left[\left(-\frac{5}{4}i\omega-\frac{15}{2}i\frac{d}{d\omega}-\frac{15}{4}i\omega\frac{d^2}{d\omega^2}\right)\left(-\frac{\pi}{2}iH_0^{(2)}(\omega x_0)\right)+\right.$$

$$\left.+\left(\frac{5}{4}+\frac{5}{4}\omega\frac{d}{d\omega}-\frac{15}{4}\frac{d^2}{d\omega^2}\right)\int_1^{\infty}\frac{x_0 s\,e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\,\frac{ds}{\sqrt{s^2-1}}-\frac{1}{4}\omega i\int_1^{\infty}\frac{x_0^2 s^2-1+x_0^2}{(x_0^2 s^2+1-x_0^2)^2}\,\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\,ds\right]\quad\ldots\quad(6,11)$$

The term still appearing beside the requisite flexural deformation must be compensated later on. The downwash formulas are all changed to reduce the paper work to a minimum in the subsequent compensation process.

The pure flexural oscillation in x or y direction could now equally be formed. But, since the downwash formula W_B^1 is very simple, while the formula for W_B^2 is much more complicated, it is advisable to postpone the linear superposition until after the compensation calculus.

REPRESENTATION OF THE TORSIONAL OSCILLATION ABOUT THE X-AXIS (ROLL OSCILLATION)

In connection with the oscillations symmetrical with respect to plane $x - z$, the unsymmetrical oscillations with respect to this plane with deformation of the disk are analyzed.

The requisite downwash shall read:

$$W = C \cdot y \cdot e^{i\nu t} \quad \ldots \quad (7,1)$$

The calculation succeeds with the pressure potential

$$\bar{\varphi}_2^1 = \bar{C}_2^1 \frac{\mu}{2} \sqrt{1-\mu^2}\sqrt{1+\eta^2}\left(3 - \frac{1}{1+\eta^2} - 3\eta \operatorname{arc\,ctg}\eta\right)\sin\vartheta \cdot e^{i\nu t} \quad \ldots \quad (7,2)$$

The downwash formula of the inner space reads:

$$W_{2,i}^{(1)} = -\frac{\bar{C}_2^1}{V}\frac{3}{4}\pi\int_{-x_0}^{x} y \cdot e^{i\nu t'}\,dx' = -\frac{\bar{C}_2^1}{V}\cdot\frac{3}{4}\pi\cdot\frac{y}{\omega i}e^{i\nu t} + \frac{\bar{C}_2^1}{V}\cdot e^{i(\nu t-\omega x)}\cdot\frac{3}{4}\pi\cdot\frac{y}{\omega i}e^{-\omega i x_0}$$

and that of the outer space

$$W_{2,a}^{(1)} = \frac{\bar{C}_2^1}{V}e^{i(\nu t-\omega x)}\frac{1}{2}\int_0^\infty\left(3 - \frac{1}{1+\eta^2} - 3\eta\operatorname{arc\,ctg}\eta\right)\frac{\sqrt{1-x_0^2}}{\sqrt{x_0^2+\eta^2}}e^{i\omega x'}\,d\eta =$$

$$= \frac{\bar{C}_2^1}{V}e^{i(\nu t-\omega x)}\frac{1}{2}\int_1^\infty\left(3 - 3x_0\sqrt{s^2-1}\operatorname{arc\,ctg}x_0\sqrt{s^2-1} - \frac{1}{x_0^2 s^2+1-x_0^2}\right)\cdot\frac{y\cdot e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\,ds;$$

or, with the previously defined abbreviations:

$$\overline{W}_{2,a}^{(1)} = \frac{3}{2}y\,\overline{W}_{1,a}^{0} - \frac{1}{2}y\int_1^\infty\frac{e^{-\omega i x_0 s}}{(x_0^2 s^2+1-x_0^2)}\cdot\frac{ds}{\sqrt{s^2-1}}.$$

Bearing in mind that

$$\int_1^\infty\frac{e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} = -i\int_1^\infty ds\int_0^\omega\frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{d\omega}{\sqrt{s^2-1}} + C$$

with the constant

$$C = \int_1^\infty\frac{1}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} = \frac{1}{y}\cdot Q_0(y),$$

the total downwash follows at:

$$W_{ll} = W_{2,i}^{(1)} + W_{2,a}^{(1)}$$

$$W_{ll} = -\frac{\bar{C}_2^1}{V}\frac{3}{4}\pi\frac{y}{\omega i}\cdot e^{i\nu t} + \frac{\bar{C}_2^1}{V}\cdot e^{i(\nu t-\omega x)}\left[-\frac{3}{4}\pi i y\cdot H_0^{(2)}(\omega\cdot x_0) - \right.$$

$$\left. -\frac{3y}{2\omega i}\int_1^\infty\frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} + \frac{i}{2}y\int_0^\omega d\omega\int_1^\infty\frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} - \frac{1}{2}Q_0(y)\right] \quad \ldots \quad (7,3)$$

It thus affords the requisite downwash (7.1) and a term to be compensated.

BENDING OSCILLATION

The last oscillation with a disk deformation of the second degree in x and y shall be represented by the downwash

$$W = E\cdot x\cdot y\cdot e^{i\nu t} \quad \ldots \quad (8,1)$$

It is solved with the pressure potential

$$\bar{\varphi}_3^2 = \bar{C}_3^2\frac{-\mu^3+\mu}{8}(1+\eta^2)\left(15 - \frac{5}{1+\eta^2} - \frac{2}{(1+\eta^2)^2} - 15\eta\operatorname{arc\,ctg}\eta\right)\sin 2\vartheta\cdot e^{i\nu t} \quad \ldots \quad (8,2)$$

The downwash formula of the inner space is:

$$W_{3,i}^{(2)} = \frac{\bar{C}_3^2}{V}\int_{-x_0}^{x} -\frac{15}{8}\pi x' y\cdot e^{i\nu t'}\,dx'$$

$$W_{3,i}^{(2)} = -\frac{\bar{C}_3^2}{V}\frac{15}{8}\pi y\left(\frac{1}{i\omega}x + \frac{1}{\omega^2}\right)e^{i\nu t} + \frac{\bar{C}_3^2}{V}e^{i(\nu t-\omega x)}\frac{15}{8}\pi y\left(-\frac{1}{i\omega}x_0 + \frac{1}{\omega^2}\right)\cdot e^{-\omega i x_0}$$

and with

$$\sin 2\vartheta = -\frac{2\sqrt{x_0^2+\eta^2}\sqrt{1-x_0^2}}{1+\eta^2}$$

that of the outer space reads:

$$W^{(2)}_{2.a} = -\frac{1}{4}\int_1^{\infty}\left(15 - \frac{5}{1 + x_0^2(s^2-1)} - \frac{2}{(x_0^2 s^2 + 1 - x_0^2)^2} - 15\, x_0\sqrt{s^2-1}\,\operatorname{arc\,ctg} x_0\sqrt{s^2-1}\right) y\, \frac{x_0 s\cdot e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\, ds$$

$$= \frac{5}{2}\, y\cdot W^1_{2.a} + \frac{1}{2}\, y \int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{(x_0^2 s^2 + 1 - x_0^2)^2}\cdot\frac{ds}{\sqrt{s^2-1}}. \quad \text{(8, 3)}$$

Also applicable is

$$\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{(x_0^2 s^2+1-x_0^2)^2}\cdot\frac{ds}{\sqrt{s^2-1}} = \frac{1}{4i\sqrt{1-x_0^2}}\int_1^{\infty}\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\left(\frac{1}{(x_0 s - i\sqrt{1-x_0^2})^2} - \frac{1}{(x_0 s + i\sqrt{1-x_0^2})^2}\right) ds =$$

$$= \frac{1}{4 i\sin\alpha}\left[\int_1^{\infty}\frac{e^{\omega\sin\alpha}}{\sqrt{s^2-1}}\,\frac{e^{-\omega i(s\cos\alpha - i\sin\alpha)}}{(s\cdot\cos\alpha - i\sin\alpha)^2}\, ds - \int_1^{\infty}\frac{e^{-\omega\sin\alpha}}{\sqrt{s^2-1}}\,\frac{e^{-\omega i(s\cos\alpha + i\sin\alpha)}}{(s\cos\alpha + i\sin\alpha)^2}\, ds\right].$$

which, after the usual differentiations and transformations, gives

$$\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{(x_0^2 s^2+1-x_0^2)^2}\cdot\frac{ds}{\sqrt{s^2-1}} =$$

$$= \frac{e^{+\omega\sin\alpha}}{4i\sin\alpha}\left(C_2^{(2)} - i\, C_2^{(1)}\omega - \int_0^{\omega}\!\!\int_*\!\!\int_1^{\infty}\frac{e^{-\bar\omega i x_0 s}}{\sqrt{s^2-1}}\, ds\cdot e^{-\bar\omega\sin\alpha}\, d\bar\omega\right) - \frac{e^{-\omega\sin\alpha}}{4i\sin\alpha}\left(C_1^{(2)} - i\, C_1^{(1)}\omega - \int_0^{\omega}\!\!\int_*\!\!\int_1^{\infty}\frac{e^{-\bar\omega i x_0 s}}{\sqrt{s^2-1}}\, ds\cdot e^{\bar\omega\sin\alpha}\, d\bar\omega\right),$$

where $C^{(1)}_{\substack{2\\1}}$ and $C^{(2)}_{\substack{2\\1}}$ denote the values computed under (4.6).

Hence

$$\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{(x_0^2 s^2+1-x_0^2)^2}\cdot\frac{ds}{\sqrt{s^2-1}} = \frac{1}{2i\sin\alpha}\Big(-Q_1\cdot\mathfrak{Sin}(\omega\sin\alpha) + i\,\frac{\pi}{2}\,P_1\cdot\mathfrak{Cof}(\omega\sin\alpha) +$$

$$+ Q_0\cdot\omega\,\mathfrak{Cof}(\omega\sin\alpha) - i\,\frac{\pi}{2}\,P_0\,\mathfrak{Sin}(\omega\sin\alpha) + \frac{\pi}{2}\, i\int_0^{\omega}\!\!\int_* H_0^{(2)}(\bar\omega\cos\alpha)\,\mathfrak{Sin}[(\omega-\bar\omega)\sin\alpha]\, d\bar\omega\Big) \quad \text{(8, 4)}$$

To obtain the requisite downwash put

$$\bar C_3^2 = \omega i \bar K_3^2$$

form

$$W_B^3 = W_3^{(2)} + \frac{5}{2}\, W_R\,\frac{\bar K_3^2}{\bar C_2^1}$$

and obtain:

$$W_B^3 = -\frac{\bar K_3^2}{V}\,\frac{15\pi}{8}\, x y\cdot e^{i\nu t} + \frac{\bar K_3^2}{V}\, e^{i(\nu t - \omega x)}\, y\left[\left(\frac{15}{4} + \frac{15}{4}\,\omega\,\frac{d}{d\omega}\right)\left(-\frac{\pi}{2}\, i H_0^{(2)}(\omega x_0)\right) - \frac{5}{4}\int_1^{\infty}\frac{e^{-\omega i x_0 s}}{x_0^2 s^2 + 1 - x_0^2}\,\frac{ds}{\sqrt{s^2-1}} +\right.$$

$$\left. + \left(\frac{5}{4}\, i\omega + \frac{15}{4i}\,\frac{d}{d\omega}\right)\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2 + 1 - x_0^2}\,\frac{ds}{\sqrt{s^2-1}} + \frac{\omega i}{2}\int_1^{\infty}\frac{x_0 s\cdot e^{-\omega i x_0 s}}{(x_0^2 s^2 + 1 - x_0^2)^2}\cdot\frac{ds}{\sqrt{s^2-1}}\right] \quad \text{(8, 5)}$$

the requisite downwash with the term to be compensated

Herewith all the downwash forms stipulated in the introduction are computed, and proceed to the potential functions of the second type to be used to effect the compensation of the superfluous terms of the downwash function of the first type.

Obviously other higher potential functions of the first type afford further disk deformations, whereby the downwash functions become continuously more complicated.

POTENTIAL FUNCTIONS OF THE SECOND TYPE

These are obtained from those of the first type by a limiting process. For this purpose the linear combination

$$\psi_n = \frac{\partial \psi_{n+1}^n}{\partial \mu} \cdot \frac{d\mu}{dc} + \frac{\partial \psi_{n+1}^n}{\partial \eta} \cdot \frac{d\eta}{dc} + (n+1)\frac{\psi_{n+1}^n}{c} \quad (9,1)$$

is formed, where c is the radius of the circular surface. The ψ_n also satisfy the potential equation.

This differentiation could, of course, be equally applied to the downwash of the potential functions of the first type for obtaining those of the second. But in this instance it is found to be simpler to compute the functions according to (9.1) and to define the related downwash, because the final formulas follow a pattern. Again the pressure potential

$$\varphi_n = \psi_n \cdot e^{i\nu t} \cdot \text{const} \quad \ldots \ldots \quad (9,2)$$

is valid, whereby ψ_n is as in the stationary case,

$$\psi_n = C_n \cdot \frac{\mu(1-\mu^2)^{\frac{n}{2}}}{(\mu^2+\eta^2)(1+\eta^2)^{\frac{n}{2}}} \begin{matrix}\cos n\vartheta \\ \sin n\vartheta\end{matrix} \text{ ist.} \quad . \quad (9,3)$$

These potential functions approach infinity at the entire disk edge as $1/\mu$.

The downwash then follows at

$$W_n = \frac{1}{V}\frac{d}{dz_{z\to 0}}\int_0^{\mu_0}\psi_n(\mu, y, z, t')\,d\mu \quad \ldots \quad (9,4)$$

Formula (9.3) can then be written again in the coordinates μ, y, z whereby $\begin{matrix}\cos n\vartheta \\ \sin n\vartheta\end{matrix}$ must be developed in a series of $\begin{matrix}\cos\vartheta \\ \sin\vartheta\end{matrix}$. Formula (9.3) gives with $\cos n\vartheta$ the symmetrical, and with $\sin n\vartheta$ the unsymmetrical, potential functions. For symmetrical functions the development then would give

$$\psi_n(\mu, y, z) = C_n\frac{\mu^{n+1}}{(\mu^2+z^2)^n}\Big\{[(1-\mu^2)(\mu^2+z^2)-\mu^2y^2]^{\frac{n-1}{2}} - \binom{n}{2}\mu^2y^2[(1-\mu^2)(\mu^2+z^2)-\mu^2y^2]^{\frac{n-1}{2}-1} + \binom{n}{4}\mu^4y^4[(1-\mu^2)(\mu^2+z^2)-\mu^2y^2]^{\frac{n-1}{2}-2} + \ldots\Big\}$$

As this expression indicates, the prediction of the downwash would necessitate integration over the surface else the total expression for the operator $\frac{d}{dz_{z\to 0}}$ becomes indeterminate. This would lead to serious integration difficulties in the nonstationary.

On the other hand the third Cartesian coordinate x provides a very simple and comprehensive expression as real part of a complex function

$$W_n = \frac{C_n}{V}\frac{d}{dz_{z\to 0}}\int_0^{\mu_0}\frac{(1-\mu^2)^n}{2x'}\left(\frac{1}{(x'+iy)^n}+\frac{1}{(x'-iy)^n}\right)e^{i\nu t'}d\mu, \quad \ldots \quad (9,5)$$

with the relation

$$\frac{x'^2+y^2}{1-\mu^2} = \frac{z^2+\mu^2}{\mu^2}$$

existing between x, y and μ.

The downwash integral gives in the stationary only functions in y alone, that is, the order of magnitude of the upper limit plays no part; in this instance μ can be disregarded with respect to unity.

After this simplification of the downwash formula (9.5), the following substitution is made:

$$u = \sqrt{1+\frac{z^2}{\mu^2\cos^2\alpha}}, \quad u_0 = \sqrt{1+\frac{z^2}{\mu_0^2\cos^2\alpha}},$$

where $y = \sin\alpha$

On putting

$$\bar{g}_n(u) = \frac{e^{i\nu t}}{(u\cos\alpha + i\sin\alpha)^n} + \frac{e^{i\nu t}}{(u\cos\alpha - i\sin\alpha)^n},$$

the downwash integral changes to

$$W_n = \frac{C_n}{V}\cdot\frac{(-1)^n}{2\cos^2\alpha}\frac{d}{dz}z\cdot\int_{\mu_0}^{\infty}\frac{\bar{g}_n(u)}{\sqrt{u^2-1}^3}du.$$

after which the differentiation with respect to z and $z \longrightarrow 0$ is readily carried out. It results in:

$$W_n = \frac{C_n}{V}\cdot\frac{(-1)^n}{2\cos^2\alpha}\left[\int_1^{\infty}\frac{\bar{g}_n(u)-\bar{g}_n(1)}{\sqrt{u^2-1}^3}du - \bar{g}_n(1) + \bar{g}_n(\infty)\right] \quad \ldots \quad (9,6)$$

Here $\bar{g}_n(\infty) = 0$, for $n > 0$.

With $\nu(t-t') = \omega(x-x')$ and

$$\bar{g}_n(u) = e^{i(\nu t-\omega x)}g_n(u)$$

the downwash formula (9.6) becomes

$$W_n = \frac{C_n}{V}\frac{(-1)^n}{2\cos^2\alpha}e^{i(\nu t-\omega x)}\left[\int_1^{\infty}\frac{g_n(u)-g_n(1)}{\sqrt{u^2-1}^3}du - g_n(1)+g_n(\infty)\right] \quad \ldots \quad (9,7)$$

Partial integration affords

$$\int_1^{\infty}\frac{g_n(u)}{\sqrt{u^2-1}^3}du = \left[-g_n(u)\cdot\frac{u}{\sqrt{u^2-1}}\right]_1^{\infty} + \int_1^{\infty}\frac{dg_n(u)}{du}\cdot\frac{u}{\sqrt{u^2-1}}du.$$

Whence (9.7) becomes

$$\overline{W}_n = \frac{1}{2}\cdot\frac{(-1)^n}{\cos^2\alpha}\left(\int_1^{\infty}\frac{dg_n(u)}{du}\cdot\frac{u}{\sqrt{u^2-1}}du - \left[(g_n(u)-g_n(1))\cdot\frac{u}{\sqrt{u^2-1}}\right]_1^{\infty} - g_n(1)+g_n(\infty)\right),$$

that is, for any $n > 0$:

$$W_n = \frac{C_0}{V} \cdot e^{i(\nu t - \omega x)} \cdot \frac{(-1)^n}{2\cos^2\alpha} \int_1^\infty \frac{d g_n}{du} \cdot \frac{u}{\sqrt{u^2-1}} \, du \quad (9,8)$$

1. $n = 0$.

With

$$\frac{d g_0}{du} = 2 i \omega \cos\alpha \, e^{+i\omega u \cos\alpha}$$

$$y = \sin\alpha \qquad x' = u \cdot \cos\alpha$$

the downwash reads

$$W_0 = \frac{C_0}{V} \cdot \frac{e^{i(\nu t - \omega x)}}{\cos^2\alpha} \int_1^\infty \frac{i\omega\cos\alpha \cdot e^{-i\omega u \cos\alpha} \cdot u}{\sqrt{u^2-1}} \, du \quad (9,9)$$

and with Hankel's function

$$\int_1^\infty \frac{e^{i\omega u \cos\alpha}}{\sqrt{u^2-1}} \, du = +\frac{\pi}{2} i H_0^{(1)}(\omega \cdot \cos\alpha)$$

(9.9) becomes

$$W_0 = \frac{C_0}{V} \cdot \frac{e^{i(\nu t - \omega x)}}{\cos^2\alpha} \cdot \omega \frac{d}{d\omega}\left(\frac{\pi}{2} i H_0^{(1)}(\omega \cdot \cos\alpha)\right). \quad (9,10)$$

Because

$$\frac{d}{dx} H_0^{(1)}(x) = -H_1^{(1)}(x)$$

the downwash formula finally reads

$$W_0 = -\frac{C_0}{V} \frac{e^{i(\nu t - \omega x)}}{\cos^2\alpha} \cdot \frac{\pi}{2} i \omega \cos\alpha \cdot H_1^{(1)}(\omega \cdot \cos\alpha). \quad (9,11)$$

2. $n = 1$.

With the abbreviating style of writing

$$g_n(u) = \frac{e^{i\omega u \cos\alpha}}{(u\cos\alpha + i\sin\alpha)^n} + \frac{e^{i\omega u \cos\alpha}}{(u\cos\alpha - i\sin\alpha)^n} = 2 \cdot R\left(\frac{e^{i\omega u \cos\alpha}}{(u\cos\alpha + i\sin\alpha)^n}\right)$$

the downwash reads according to (9.8):

$$W_1 = \frac{1}{\cos\alpha} \int_1^\infty R\left(\frac{e^{i\omega u \cos\alpha}}{(u\cos\alpha + i\sin\alpha)^2}\right) \frac{u}{\sqrt{u^2-1}} \, du - \frac{i\omega}{\cos\alpha} \int_1^\infty R\left(\frac{e^{i\omega u \cos\alpha}}{u\cos\alpha + i\sin\alpha}\right) \frac{u}{\sqrt{u^2-1}} \, du \quad (9,12)$$

and with

$$G_1^{(n)} = \int_1^\infty \frac{e^{i\omega u \cos\alpha}}{(u\cos\alpha + i\sin a)^n} \frac{du}{\sqrt{u^2-1}}$$

$$G_2^{(n)} = \int_1^\infty \frac{e^{i\omega u \cos\alpha}}{(u\cos\alpha - i\sin\alpha)^n} \frac{du}{\sqrt{u^2-1}}$$

the integral can be expressed with

$$\int_1^\infty \frac{u \cdot e^{\omega i u \cos\alpha}}{u\cos\alpha + i\sin\alpha} \cdot \frac{du}{\sqrt{u^2-1}} = \frac{1}{\cos\alpha}\left(\frac{\pi}{2} i H_0^{(1)}\right) - \frac{i\sin\alpha}{\cos\alpha} \int_1^\infty \frac{e^{i\omega u \cos\alpha}}{u\cos\alpha + i\sin\alpha} \cdot \frac{du}{\sqrt{u^2-1}}$$

$$\int_1^\infty \frac{u \cdot e^{\omega i u \cos\alpha}}{u\cos\alpha - i\sin\alpha} \cdot \frac{du}{\sqrt{u^2-1}} = \frac{1}{\cos\alpha}\left(\frac{\pi}{2} i H_0^{(1)}\right) + \frac{i\sin\alpha}{\cos\alpha} \int_1^\infty \frac{e^{\omega i u \cos\alpha}}{u\cos\alpha - i\sin\alpha} \cdot \frac{du}{\sqrt{u^2-1}}$$

$$\int_1^\infty R\left(\frac{e^{i\omega u \cos\alpha}}{(u\cos\alpha + i\sin\alpha)^2}\right) \cdot \frac{u \cdot du}{\sqrt{u^2-1}} = \frac{1}{\cos\alpha} \cdot \frac{G_1^{(1)} + G_2^{(1)}}{2} - i \frac{\sin\alpha}{\cos\alpha} \cdot \frac{G_1^{(2)} - G_2^{(2)}}{2}.$$

The general calculation of the functions $G_1^{(n)}$ and $G_2^{(n)}$ is carried out by the method employed in the preceding chapters and therefore not discussed in detail. It results in:

$$\left.\begin{aligned} G_1^{(n)} &= e^{\omega \sin\alpha}\left(C_1^{(n)} + i \int_0^\omega G_1^{(n-1)} e^{-\overline{\omega}\sin\alpha} \, d\overline{\omega}\right) \\ G_2^{(n)} &= e^{-\omega \sin\alpha}\left(C_2^{(n)} + i \int_0^\omega G_2^{(n-1)} e^{+\overline{\omega}\sin\alpha} \, d\overline{\omega}\right) \end{aligned}\right\} \quad (9,13)$$

where $G_1^{(o)}$ denotes Hankel's function $\frac{\pi}{2} i H_o^{(1)}$. In this manner the individual G functions can be successively built up with Hankel's function and solved.

The constants $C_1^{(n)}$ are defined according to formula (4.6).

That means:

$$\begin{aligned} G_{\substack{1\\2}}^{(2k+1)} = e^{\pm\omega\sin\alpha}(-1)^k \Big[& \frac{\pi}{2} P_{2k} \mp i Q_{2k} + \\ & + i\omega\left(Q_{2k-1} \pm i\frac{\pi}{2} P_{2k-1}\right) - \\ & - \frac{i^2}{2!}\omega^2\left(\frac{\pi}{2} P_{2k-2} \mp i Q_{2k-2}\right) - \\ & - \frac{i^3}{3!}\omega^3\left(Q_{2k-3} \pm i\frac{\pi}{2} P_{2k-3}\right) \cdots\cdots \\ & \cdots\cdots + (-1)^k \cdot i^{2k+1} \underbrace{\int_0^\omega \int \cdots \int_*}_{2k+1} \frac{\pi}{2} i H_0^{(1)} e^{\mp\overline{\omega}\sin\alpha} \, d\overline{\omega} \Big] \end{aligned}$$

$$\begin{aligned} G_{\substack{1\\2}}^{(2k)} = e^{\pm\omega\sin\alpha}(-1)^k \Big[& Q_{2k-1} \pm i\frac{\pi}{2} P_{2k-1} - \\ & - i\omega\left(\frac{\pi}{2} P_{2k-2} \mp i Q_{2k-2}\right) - \\ & - \frac{i^2}{2!}\omega^2\left(Q_{2k-3} \pm i\frac{\pi}{2} P_{2k-3}\right) + \cdots\cdots + \\ & + (-1)^k \cdot i^{2k} \underbrace{\int_0^\omega \int \cdots \int_*}_{2k} \frac{\pi}{2} i H_0^{(1)} e^{\mp\overline{\omega}\sin\alpha} \, d\overline{\omega} \Big] \end{aligned} \quad (9,14)$$

The rest of the downwash formulas are built up with:

$$\int_1^\infty R\left(\frac{e^{\omega i u \cos\alpha}}{(u\cos\alpha+i\sin\alpha)^n}\right)\frac{u\,du}{\sqrt{u^2-1}}=\frac{1}{\cos\alpha}\cdot\frac{G_1^{(n-1)}+G_2^{(n-1)}}{2}-i\frac{\sin\alpha}{\cos\alpha}\cdot\frac{G_1^{(n)}-G_2^{(n)}}{2}\quad\quad(9,15)$$

Whence the downwash (9.12) can be expressed as

$$W_1=\frac{1}{\cos^2\alpha}\left[\frac{\pi}{2}P_0\cdot\mathfrak{Cof}(\omega\cdot y)-iQ_0\cdot\mathfrak{Sin}(\omega y)+i\int_0^\omega\frac{\pi}{2}iH_0^{(1)}\mathfrak{Cof}[(\omega-\varpi)y]\,d\varpi+\right.$$

$$+i\sin\alpha\left\{+i\frac{\pi}{2}P_1\mathfrak{Cof}(\omega y)+Q_1\cdot\mathfrak{Sin}(\omega y)-i\omega\left(\frac{\pi}{2}P_0\mathfrak{Sin}(\omega y)-iQ_0\cdot\mathfrak{Cof}(\omega y)\right)-i^2\int_0\!\!\int_*\frac{\pi}{2}iH_0^{(1)}\mathfrak{Sin}[(\omega-\varpi)y]\,d\varpi\right\}\Bigg]-$$

$$-\frac{i\omega}{\cos^2\alpha}\left[\frac{\pi}{2}iH_0^{(1)}-i\sin\alpha\left\{\frac{\pi}{2}P_0\cdot\mathfrak{Sin}(\omega y)-iQ_0\mathfrak{Cof}(\omega y)+i\int_0^\omega\frac{\pi}{2}iH_0^{(1)}\mathfrak{Sin}[(\omega-\varpi)\cdot y]\,d\varpi\right\}\right]$$

Now the relation (reference 3)

$$(y^2-1)\frac{dP_n}{dy}=n\,(y\,P_n-P_{n-1}),\quad\quad(9,16)$$

exists between the spherical functions P_n which also holds for Q_n. Herewith the above downwash formula becomes:

$$\overline{W}_1=\frac{\pi}{2}\frac{dP_1}{dy}\mathfrak{Cof}(\omega y)-i\frac{dQ_1}{dy}\cdot\mathfrak{Sin}(\omega y)+\frac{1}{\cos^2\alpha}\left\{i\int_0^\omega\frac{\pi}{2}iH_0^{(1)}\mathfrak{Cof}[(\omega-\varpi)y]\,d\varpi-\right.$$

$$\left.-i^2y\int_0^\omega\!\!\int\frac{\pi}{2}iH_0^{(1)}\mathfrak{Sin}[(\omega-\varpi)y]\,d\varpi-i\omega\frac{\pi}{2}iH_0^{(1)}+i^2\omega y\int_0^\omega\frac{\pi}{2}iH_0^{(1)}\mathfrak{Sin}[(\omega-\varpi)y]\,d\varpi\right\}$$

Further transformation by writing

$$\int_0^\omega\!\!\int_*H_0^{(1)}\mathfrak{Sin}[(\omega-\varpi)y]\,d\varpi=\omega\int_0^\omega H_0^{(1)}\cdot\mathfrak{Sin}[(\omega-\varpi)y]\,d\varpi-\int_0^\omega\varpi H_0^{(1)}\cdot\mathfrak{Sin}[(\omega-\varpi)y]\,d\varpi=$$

$$=+\omega\int_0^\omega H_0^{(1)}\mathfrak{Sin}[(\omega-\varpi)y]\,d\varpi+\frac{\omega}{y}\cdot H_0^{(1)}-\frac{1}{y}\int_0^\omega\left(H_0^{(1)}+\omega\frac{d}{d\omega}H_0^{(1)}\right)\mathfrak{Cof}[(\omega-\varpi)y]\,d\varpi,$$

the application of the relation $\frac{d}{d\omega}H_0^{(1)}=-\cos\alpha\;H_1^{(1)}$ before (9.11) affords the final formula

$$W_1=\frac{C_1}{V}e^{i(\nu t-\omega x)}\left(\frac{\pi}{2}\frac{dP_1}{dy}\mathfrak{Cof}(\omega y)-i\frac{dQ_1}{dy}\cdot\mathfrak{Sin}(\omega y)+\frac{i}{\cos\alpha}\int_0^\omega\frac{\pi}{2}i\varpi H_1^{(1)}(\varpi\cdot\cos\alpha)\mathfrak{Cof}[(\omega-\varpi)y]\,d\varpi\right)\quad\quad(9,17)$$

<u>3. $n=2$.</u>

The downwash reads:

$$W_2=\frac{C_2}{V}\cdot\frac{e^{i(\nu t-\omega x)}}{\cos^2\alpha}\int_1^\infty\frac{dg_2(u)}{du}\cdot\frac{u}{\sqrt{u^2-1}}\,du\quad\quad(9,18)$$

with

$$\frac{dg_2(u)}{du}=-R\left(\frac{2\cos\alpha\cdot e^{i\omega u\cos\alpha}}{(u\cos\alpha+i\sin\alpha)^3}\right)+R\left(\frac{i\omega\cos\alpha\,e^{i\omega u\cos\alpha}}{(u\cos\alpha+i\sin\alpha)^2}\right).$$

$$W_2=-\frac{2}{\cos^2\alpha}\left[-Q_1\mathfrak{Cof}(\omega y)-i\frac{\pi}{2}P_1\cdot\mathfrak{Sin}(\omega y)+i\omega\left\{\frac{\pi}{2}P_0\mathfrak{Cof}(\omega y)-iQ_0\mathfrak{Sin}(\omega y)\right\}+\right.$$

$$+i^2\int_0^\omega\!\!\int_*\frac{\pi}{2}iH_0^{(1)}\mathfrak{Cof}[(\omega-\varpi)y]\,d\varpi+i\sin\alpha\left\{\frac{\pi}{2}P_2\mathfrak{Sin}(\omega y)-iQ_2\mathfrak{Cof}(\omega y)+\right.$$

$$+i\omega\left(Q_1\cdot\mathfrak{Sin}(\omega y)+i\frac{\pi}{2}P_1\mathfrak{Cof}(\omega y)\right)-\frac{i^2}{2!}\omega^2\left(\frac{\pi}{2}P_0\mathfrak{Sin}(\omega y)-iQ_0\mathfrak{Cof}(\omega y)\right)-$$

$$\left.\left.-i^3\int_0^\omega\!\!\int\!\!\int_*\frac{\pi}{2}iH_0^{(1)}\mathfrak{Sin}[(\omega-\varpi)y]\,d\varpi\right\}\right]+\frac{i\omega}{\cos^2\alpha}\left[\frac{\pi}{2}P_0\mathfrak{Cof}(\omega y)-iQ_0\mathfrak{Sin}(\omega y)+\right.$$

$$+i\int_0^\omega\frac{\pi}{2}iH_0^{(1)}\mathfrak{Cof}[(\omega-\varpi)y]\,d\varpi+i\sin\alpha\left\{Q_1\mathfrak{Sin}(\omega y)+i\frac{\pi}{2}P_1\mathfrak{Cof}\,\omega y-\right.$$

According to (9.14) and (9.15) the downwash (9.18) then reads:

$$-i\omega\left(\frac{\pi}{2}P_0\,\mathfrak{Sin}\,(\omega y)-iQ_0\,\mathfrak{Cof}\,(\omega y)\right)-i^2\int_0^{\omega}\!\!\int_*\frac{\pi}{2}\,i\,H_0^{(1)}\cdot\mathfrak{Sin}\,[(\omega-\varpi)\,y]\,d\varpi\Big\}\Big].$$

The existing multiple integrals are reformed by repeated partial integration as follows:

$$\underbrace{\int_0^{\omega}\!\!\int\cdots\int_*}_{n}\frac{\pi}{2}\,i\,H_0^{(1)}\,\begin{matrix}\mathfrak{Cof}\,[(\omega-\varpi)\,y]\\ \mathfrak{Sin}\,[(\omega-\varpi)\,y]\end{matrix}\,d\varpi=\frac{1}{(n-1)!}\int_0^{\omega}(\omega-\varpi)^{n-1}\frac{\pi}{2}\,i\,H_0^{(1)}\,\begin{matrix}\mathfrak{Cof}\,[(\omega-\varpi)\,y]\\ \mathfrak{Sin}\,[(\omega-\varpi)\,y]\end{matrix}\,d\varpi.$$

Conformably to (9.15) this can be consolidated in the downwash function W_n in the following manner:

$$\frac{n}{(n-1)!}\int_0^{\omega}(\omega-\varpi)^{n-1}\frac{\pi}{2}\,i\,H_0^{(1)}\,\mathfrak{Cof}\,[(\omega-\varpi)\,y]\,d\varpi-\frac{\omega}{(n-2)!}\int_0^{\omega}(\omega-\varpi)^{n-2}\frac{\pi}{2}\,i\,H_0^{(1)}\,\mathfrak{Cof}\,[(\omega-\varpi)\,y]\,d\varpi+$$

$$+\frac{n\,y}{n!}\int_0^{\omega}(\omega-\varpi)^{n}\frac{\pi}{2}\,i\,H_0^{(1)}\,\mathfrak{Sin}\,[(\omega-\varpi)\,y]\,d\varpi-\frac{\omega\,y}{(n-1)!}\int_0^{\omega}(\omega-\varpi)^{n-1}\frac{\pi}{2}\,i\,H_0^{(1)}\,\mathfrak{Sin}\,[(\omega-\varpi)\,y]\,d\varpi=$$

$$=\frac{1}{(n-1)!}\int_0^{\omega}(\omega-\varpi)^{n-1}\left(-\varpi\frac{\pi}{2}\,i\,\frac{d}{d\varpi}H_0^{(1)}\right)\mathfrak{Cof}\,[(\omega-\varpi)\,y]\,d\varpi,\quad\ldots\ldots\ldots\ldots \quad (9,19)$$

if the last two integrals on the left-hand side are combined and transformed through partial integration, by putting

$$\mathfrak{Sin}\,[(\omega-\varpi)\,y]\,d\varpi=\frac{-1}{y}\,d\,\mathfrak{Cof}\,[(\omega-\varpi)\,y]$$

Further allowance for (9.16) and for the fact that

$$\frac{d}{d\omega}H_0^{(1)}=-\cos\alpha\,H_1^{(1)}$$

affords collectively:

$$W_2=\frac{C_2}{V}\cdot e^{i(\nu t-\omega x)}\left[\frac{dQ_2}{dy}\cdot\mathfrak{Cof}\,(\omega y)+i\,\frac{\pi}{2}\,\frac{dP_2}{dy}\cdot\mathfrak{Sin}\,(\omega y)-\omega\left(\frac{dQ_1}{dy}\,\mathfrak{Sin}\,(\omega y)+i\,\frac{\pi}{2}\,\frac{dP_1}{dy}\cdot\mathfrak{Cof}\,(\omega y)\right)+\right.$$

$$\left.+\frac{1}{\cos\alpha}\int_0^{\omega}(\omega-\varpi)\,\varpi\,\frac{\pi}{2}\,i\,H_1^{(1)}\,(\varpi\cos\alpha)\,\mathfrak{Cof}\,[(\omega-\varpi)\,y]\,d\varpi\right]\ \ldots\ldots\ldots\ldots \quad (9,20)$$

The general construction of the n^{th} downwash formula is readily apparent from the foregoing.

It is

$$W_n=\frac{C_n}{V}\,e^{i(\nu t-\omega x)}\cdot(-1)^{n+1}\left[\frac{i^n}{\cos\alpha}\int_0^{\omega}\frac{(\omega-\varpi)^{n-1}}{(n-1)!}\,\varpi\,\frac{\pi}{2}\,i\,H_1^{(1)}\cdot\mathfrak{Cof}\,[(\omega-\varpi)\,y]\,d\varpi+\right.$$

$$+\frac{i^{n-1}}{(n-1)!}\left(\frac{\pi}{2}\,\frac{dP_1}{dy}\,\mathfrak{Cof}\,(\omega y)-i\,\frac{dQ_1}{dy}\cdot\mathfrak{Sin}\,(\omega y)\right)\cdot\omega^{n-1}-\frac{i^{n-2}}{(n-2)!}\left(\frac{dQ_2}{dy}\,\mathfrak{Cof}\,(\omega y)+i\,\frac{\pi}{2}\,\frac{dP_2}{dy}\cdot\mathfrak{Sin}\,(\omega y)\right)\cdot\omega^{n-2}$$

$$\left.-\frac{i^{n-3}}{(n-3)!}\left(\frac{\pi}{2}\,\frac{dP_3}{dy}\,\mathfrak{Cof}\,(\omega y)-i\,\frac{dQ_3}{dy}\cdot\mathfrak{Sin}\,(\omega y)\right)\cdot\omega^{n-3}+\frac{i^{n-4}}{(n-4)!}\left(\frac{dQ_4}{dy}\,\mathfrak{Cof}\,(\omega y)+i\,\frac{\pi}{2}\,\frac{dP_4}{dy}\cdot\mathfrak{Sin}\,(\omega y)\right)\cdot\omega^{n-4}+\ldots\right]$$

$$\ldots\ (9,21)$$

These symmetrical downwash functions of the second type, which are even in y, are suitable for the compensation of the terms in the symmetrical downwash functions of the first type (4.7), (5.4), (6.6), and (6.11) which carry the factor $e^{i(\nu t-\omega x)}$ and are for that reason unsuitable for the representation of the airfoil motion.

For the compensation of the corresponding terms of the unsymmetrical downwash functions of the second type (7.3), (8.5) the unsymmetrical downwash functions of the second type which are odd in y are derived similarly.

The potential unsymmetrical to the $x-z$ plane reads

$$\varphi_n = \bar{C}_n \frac{\mu (1-\mu^2)^{n/2}}{(\mu^2+\eta^2)(1+\eta^2)^{n/2}} \sin n\vartheta \, e^{i\nu t} \quad \ldots \ldots \quad (9,22)$$

For computing the downwash the similar complex style of writing is resorted to

$$W_n = \frac{\bar{C}_n}{V}(-i)(-1)^n \frac{d}{dz}_{z\to 0} \frac{1}{2}\int_0^{\mu_0}\left(\frac{(1-\mu^2)^n}{(x'+iy)^n} - \frac{(1-\mu^2)^n}{(x'-iy)^n}\right)\frac{e^{i\nu t'}}{x'}\, d\mu.$$

Application of the same transformations used in the prediction of the symmetrical downwash function, and putting

$$g_n(u) = e^{i\omega u\cos\alpha}\left(\frac{1}{(u\cdot\cos\alpha + i\sin\alpha)^n} - \frac{1}{(u\cdot\cos\alpha - i\sin\alpha)^n}\right) = 2I\left(\frac{e^{i\omega u\cos\alpha}}{(u\cdot\cos\alpha+i\sin\alpha)^n}\right),$$

yields the corresponding downwash integral at

$$W_n = \frac{\bar{C}_n}{V}\cdot(-i)\cdot(-1)^n\cdot\frac{e^{i(\nu t-\omega x)}}{2\cos^2\alpha}\int_1^\infty \frac{dg_n(u)}{du}\cdot\frac{u}{\sqrt{u^2-1}}\,du \quad \ldots \ldots \quad (9,23)$$

The subsequent calculation of the integrals proceeds in the same manner as for the symmetrical downwash functions. The general term for the unsymmetrical downwash functions then reads:

$$\begin{aligned} W_n = \frac{C_n}{V} e^{i(\nu t-\omega x)}(-1)^n &\left[\frac{i^{n+1}}{\cos\alpha}\int_0^\omega \frac{(\omega-\bar{\omega})^{n-1}}{(n-1)!}\bar{\omega}\frac{\pi}{2} iH_1^{(1)}\mathfrak{Sin}[(\omega-\bar{\omega})y]\,d\bar{\omega} + \right. \qquad (9,24)\\ &+\frac{i^n}{(n-1)!}\omega^{n-1}\left(\frac{\pi}{2}\frac{dP_1}{dy}\mathfrak{Sin}(\omega\cdot y) - i\frac{dQ_1}{dy}\mathfrak{Cof}(\omega y)\right) - \frac{i^{n-1}}{(n-2)!}\omega^{n-2}\left(\frac{dQ_2}{dy}\mathfrak{Sin}(\omega\cdot y) + i\frac{\pi}{2}\frac{dP_2}{dy}\mathfrak{Cof}(\omega y)\right) - \\ &\left. -\frac{i^{n-2}}{(n-3)!}\omega^{n-3}\left(\frac{\pi}{2}\frac{dP_3}{dy}\mathfrak{Sin}(\omega\cdot y) - i\frac{dQ_3}{dy}\mathfrak{Cof}(\omega y)\right) + \frac{i^{n-3}}{(n-4)!}\omega^{n-4}\left(\frac{dQ_4}{dy}\mathfrak{Sin}(\omega\cdot y) + i\frac{\pi}{2}\frac{dP_4}{dy}\mathfrak{Cof}(\omega y)\right) + \ldots\right] \end{aligned}$$

It is readily seen that this method allows a continuous and lucid presentation of the downwash functions of the second type, and furthermore, that with $\omega \to 0$ the same stationary values of the downwash functions are obtained as in the cited report by Kinner.

THE FLOW-OFF CONDITION (cf. reference 1)

The condition for smooth flow-off from the trailing edge implies the disappearance of the values of the potential functions for

$$\mu = 0 \qquad \eta = 0 \qquad -\frac{\pi}{2} < \varphi < +\frac{\pi}{2}$$

This condition is satisfied for every potential function of the first type, because they disappear on the entire disk edge. For those of the second type it had been stated previously that their values approached infinity as $1/\mu$.

For the linear superposition of the compensation process an infinite sum of potential functions of the second type is formed. Since this sum must approach zero at the trailing edge, the conditional equation is:

$$\sum_{k=0}^{\infty} C_{2k} \cos 2k\varphi + \sum_{\lambda=0}^{\infty} C_{2\lambda+1} \cos(2\lambda+1)\varphi = 0 \qquad (10,1)$$

To evaluate this relation analytically, multiply (10,1) by $\cos(2\lambda + 1)\varphi$, and integrate over φ from 0 to $\frac{\pi}{2}$.

Then

$$C_{2\lambda+1} = \frac{4}{\pi} (-1)^{\lambda+1} (2\lambda+1) \sum_{k=0}^{\infty} \frac{(-1)^k C_{2k}}{(2\lambda+1)^2 - (2k)^2} \qquad (10,2)$$

For unsymmetrical potential functions of the second type the condition of smooth flow-off reads:

$$\sum_{k=0}^{\infty} \bar{C}_{2k} \sin 2k\varphi + \sum_{\lambda=0}^{\infty} \bar{C}_{2\lambda+1} \sin(2\lambda+1)\varphi = 0 \qquad (10,3)$$

Multiplication of (10,3) by $\sin 2K\varphi$ followed by integration gives

$$\bar{C}_{2k} = -\frac{4}{\pi} (-1)^k \, 2k \sum_{\lambda=0}^{\infty} \frac{(-1)^\lambda \bar{C}_{2\lambda+1}}{(2\lambda+1)^2 - (2k)^2}$$

COMPENSATION CALCULUS

The general expression for the downwash function of the first type is according to (3,1):

$$W = f(x,y) e^{i\nu t} + e^{i(\nu t - \omega x)} g(y) \qquad (11,1)$$

where $f(x, y) e^{i\nu t}$ exactly represents the requisite downwash. The downwash functions of the second type are represented by

$$W_n = C_n h_n(y) e^{i(\nu t - \omega x)} \qquad (11,2)$$

Equations (11,1) and (11,2) are linearly superposed, so that the following is complied with

$$g(y) + \sum_{n=0}^{\infty} C_n h_n(y) = 0 \tag{11,3}$$

whereby $g(y)$ and $h_n(y)$ both must either be symmetrical or unsymmetrical downwash functions, respectively, the $h_n(y)$ are generally read from (9,2) and (9,24). The odd C_n $(n = 2\lambda + 1)$ and even C_n $(n = 2k)$, respectively, follow from the flow-off condition treated in the preceding section. For the stationary case $(\omega \rightarrow 0)$ the orthogonal characteristics of the spherical functions is employed in the analytical problem, resulting in infinitely many equations for infinitely many unknowns C_n (reference 1).

In the nonstationary $(\omega \neq 0)$, no orthogonal functions to $h_n(y)$ are known.

In view of the repeated occurrence of the terms in $\cos^{2n}\alpha$ in (9,21) and (9,24) the orthogonality of the trigonometric functions is employed.

The problem could be solved for symmetrical $g(y)$ and $h_n(y)$ (11,3) by multiplying with $\cos 2k\alpha$ and integrating from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

But $\frac{dQ_n}{dy}$ has the general form

$$\frac{dQ_n}{dy} = \frac{a_1(y)}{1 - y^2} + b_1(y) \quad \ln \frac{1 + y}{1 - y}$$

where $a_1(y)$, and $b_1(y)$ are polynomials in y. Since the denominator $(1 - y^2)$ would impede the integration over α, multiply (11,3) by $\cos^2\alpha \times \cos 2k\alpha$ and then integrate. It affords infinitely many inhomogeneous linear equations for the still unknown C_{2k}.

The unsymmetrical downwash functions of the first and second type are largely provided with the factor $y = \sin\alpha$ but, for the rest, even again in y. So in order to enable the use of the integrations over α, multiply in this case (11,3) by $\frac{\cos^2\alpha}{\sin\alpha}$ $\cos 2k\alpha$ and integrate over α from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$.

The result is again an infinite number of equations for the still unknowns $\bar{C}_{2\lambda+1}$.

The constants C_n and $\bar{C}_n$, respectively, are, then, functions of ω only.

PRESSURE DISTRIBUTION, LIFT, AND MOMENTS

From the relation

$$\underline{b} + \frac{1}{\rho} \operatorname{grad} p = 0 \tag{12,1}$$

imposed by Euler's equation, and $\underline{b} = \operatorname{grad} \varphi$ follows the equation

$$\rho \varphi = p_\infty - p \tag{12,2}$$

where p_∞ is the static pressure at infinity. With it the resultant pressure per unit of surface in a point at the airfoil for short pressure jump, is:

$$\Pi (x, y, t) \equiv p_u - p_{ob} = \rho (\varphi_{ob} - \varphi_u) \tag{12,3}$$

where $\eta = 0$ in the potential functions. Since the values of the potential functions on the upper and lower surface are inversely equal, equation (12,3) can be written

$$\Pi (x, y, t) = p_u - p_{ob} = 2 \rho \varphi_{ob} \tag{12,4}$$

Hence, with the values of the coefficients of the potential function defined by compensation calculus, the pressure distribution can be solved with (12,4) as it will be achieved numerically in part II.

The lift follows at

$$A = \iint \Pi \ F d \tag{12,5}$$

with the surface integral extending over the disk of radius c.

With

$$dF = c^2 \mu d\mu d\varphi$$

(12,5) becomes

$$A = c^2 \int_{\varphi=0}^{2\pi} \int_{\mu=0}^{1} \Pi \, \mu d\mu d\varphi \tag{12,6}$$

Bearing in mind the relation

$$\int_0^{2\pi} \cos n\varphi d\varphi = 0 \qquad \int_0^{2\pi} \sin n\varphi d\varphi = 0 \qquad n = 1, 2, 3 \ldots$$

it is evident that

$$A_1^0 = \frac{4}{3} \pi \rho c^2 C_1^0 e^{i\upsilon t} \qquad A_0 = 4\pi\rho c^2 C_0 e^{i\upsilon t} \tag{12,7}$$

while for all other φ_n^m and φ_n the lift disappears, so that the calculation of the total lift becomes very simple. Thus the total lift will scarcely depend upon the convergence of the C_n coefficients. This does not hold true for the pressure distribution.

The pitching moment of the lift forces about the y-axis reads:

$$M = \iint \Pi \, x dF = c^3 \int_{\varphi=0}^{2\pi} \int_{\mu=0}^{1} \Pi \mu \sqrt{1 - \mu^2} \cos\varphi \, d\mu d\varphi \tag{12,8}$$

With the relations given in the lift calculation, it affords in this instance

$$M_2^1 = \frac{4}{15} \pi \rho c^3 C_2^1 e^{i\upsilon t} \qquad M_1 = \frac{4}{3} \pi \rho c^3 C_1 e^{i\upsilon t} \tag{12,9}$$

while the moment for all other potential functions φ_n^m and φ_n vanishes.

Naturally, this applies only to symmetrical potential functions. If they are unsymmetrical to the x, y-plane, the formulas (12,9) indicate the amount of the rolling moments, in which case the corresponding $\underline{C}$ of the unsymmetrical potential functions must be inserted.

PART II

By K. Krienes and Th. Schade

This part deals with the aerodynamic coefficients for downwash distributions which are arbitrary functions of the second degree, that is, in the range of the reduced frequencies $\omega = 0$ to 2.0. The pressure distribution is secured also in several cases. The numerical results are illustrated by graphical representation.

INTRODUCTION

As already explained in part I, the pressure or acceleration potential is put down as sum of potential functions of the first and second type. The functions of the first type are obtained by a separation theorem from Laplace's differential equation $\Delta\varphi = 0$ after introduction of elliptic coordinates

$$\varphi_n^m(\mu, \eta, \vartheta) = \frac{1}{i^{n-m+1}} \cdot \frac{(n-m)!}{(n+m)!} C_n^m P_n^m(\mu) Q_n^m(i\eta) \cdot \frac{\cos m\vartheta}{\sin m\vartheta} e^{i\nu t} \qquad (1,1)$$

The potential functions of the second type are obtained by differentiations of φ_{n+1}^n with respect to the base circle radius c for constants x, y, and z.

$$\varphi_n = C_n \frac{\mu(1-\mu^2)^{\frac{n}{2}}}{(\mu^2+\eta^2)(1+\eta^2)^{\frac{n}{2}}} \frac{\cos n\vartheta}{\sin n\vartheta} e^{i\nu t} \qquad (1,2)$$

The downwash velocities induced by these potential functions at the lifting surface – termed downwash for short – have been calculated in part I and are briefly restated here.

The potential $$\varphi_s = i\omega\varphi_1^0 \qquad (1,3)$$

yields $$W_s = -\frac{\pi}{2}\frac{1}{V}e^{i\nu t} + \frac{1}{V}e^{i(\nu t-\omega x)}\left[\frac{\pi}{2}\omega H_0^{(2)}(\omega x_0) + \int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2 + 1 - x_0^2}\frac{ds}{\sqrt{s^2-1}}\right] \qquad (1,4)$$

$$\varphi_k = i\omega\varphi_2^1 + \frac{3}{2}\varphi_1^0 \qquad (1,5)$$

yields

$$W_k = -\frac{3\pi}{4}\frac{1}{V}x\cdot e^{i\nu t} + \frac{1}{V}e^{i(\nu t-\omega x)}\left[\left(-\frac{3}{2} - \frac{3}{2}\omega\frac{d}{d\omega}\right)\cdot\frac{\pi}{2}iH_0^{(2)}(\omega x_0) + \right.$$
$$\left. + \frac{i}{2}\left(\omega - 3\frac{d}{d\omega}\right)\int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2 + 1 - x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}}\right] \qquad (1,6)$$

$$\varphi_3^1 = i\omega\varphi_3^0 - 5\varphi_2^1 - \frac{3}{2}i\omega\varphi_1^0 \qquad (1,7)$$

yields

$$W_2^1 = \frac{15\pi}{8}\frac{1}{V}(x^2+y^2)\,e^{i\nu t} + \frac{1}{V}e^{i(\nu t-\omega x)}\left[-\frac{5}{2}\cdot\frac{\pi}{2}\,\omega H_0^{(2)}(\omega x_0) - \frac{5}{2}\int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}}\right]. \quad (1,8)$$

(The term to be compensated (cf. Compensation Calculation) is, up to the factor $-\frac{5}{2}$, the same as for the downwash function W_s (1.4)

$$\varphi_2^2 = i\omega\,\varphi_s^2 + \frac{5}{2}\varphi_2^1 \quad (1,9)$$

yields

$$W_2^2 = \frac{15}{16}\pi\cdot\frac{1}{V}(y^2-x^2)\,e^{i\nu t} + \frac{1}{V}e^{i(\nu t-\omega x)}\left[\left(\frac{5}{4}i\omega + \frac{15}{2}i\frac{d}{d\omega} + \frac{15}{4}i\omega\frac{d^2}{d\omega^2}\right)\left(\frac{\pi}{2}iH_0^{(2)}(\omega x_0)\right)\right.$$
$$+\left(\frac{5}{4}+\frac{5}{4}\omega\frac{d}{d\omega}-\frac{15}{4}\frac{d^2}{d\omega^2}\right)\int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}}$$
$$\left.-\frac{1}{4}i\omega\int_1^\infty \frac{x_0^2 s^2-1+x_0^2}{(x_0^2 s^2+1-x_0^2)^2}\cdot\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\,ds\right] \quad (1,10)$$

of the potential functions antisymmetrical in y

yields

$$\varphi_s = i\omega\,\overline{\varphi}_2^1 \quad (1,11)$$

$$W_s = -\frac{3\pi}{4}\cdot\frac{1}{V}\cdot y\,e^{i\nu t} + \frac{1}{V}e^{i(\nu t-\omega x)}\left[\frac{3}{2}y\cdot\omega\cdot\frac{\pi}{2}H_0^{(2)}(\omega x_0)\right.$$
$$\left.-\frac{3}{2}y\int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} - \frac{y}{2}i\omega\int_1^\infty \frac{e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}}\right] \quad (1,12)$$

and

$$\varphi_s^2 = i\omega\,\overline{\varphi}_s^2 + \frac{5}{2}\overline{\varphi}_2^1 \quad (1,13)$$

yields

$$W_s^2 = -\frac{15\pi}{8}\frac{1}{V}x\cdot y\cdot e^{i\nu t} + \frac{1}{V}e^{i(\nu t-\omega x)}\cdot y\cdot\left[\left(\frac{15}{4}+\frac{15}{4}\omega\frac{d}{d\omega}\right)\left(-\frac{\pi}{2}iH_0^{(2)}(\omega x_0)\right)\right.$$
$$-\frac{5}{4}\int_1^\infty \frac{e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} + \left(\frac{5}{4}i\omega+\frac{15}{4i}\frac{d}{d\omega}\right)\int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}}$$
$$\left.+\frac{i\omega}{2}\int_1^\infty \frac{x_0 s\; e^{-\omega i x_0 s}}{(x_0^2 s^2+1-x_0^2)^2}\cdot\frac{ds}{\sqrt{s^2-1}}\right] \quad (1,14)$$

Hereby

$$\int_1^\infty \frac{x_0 s\cdot e^{-\omega i x_0 s}}{x_0^2 s^2+1-x_0^2}\cdot\frac{ds}{\sqrt{s^2-1}} = iQ_0(y)\,\mathfrak{Sin}(\omega y) + \frac{\pi}{2}P_0(y)\,\mathfrak{Cof}(\omega y)$$
$$-\frac{\pi}{2}\int_0^\infty H_0^{(2)}(\varpi x_0)\,\mathfrak{Cof}[(\omega-\varpi)y]\,d\varpi \quad (1,15)$$

$$\int_1^\infty \frac{x_0^2 s^2-1+x_0^2}{(x_0^2 s^2+1-x_0^2)^2}\cdot\frac{e^{-\omega i x_0 s}}{\sqrt{s^2-1}}\,ds = -Q_1(y)\,\mathfrak{Cof}(\omega y) + \frac{\pi}{2}iP_1(y)\,\mathfrak{Sin}(\omega y)$$
$$-\frac{\pi}{2}i\omega P_0(y)\,\mathfrak{Cof}(\omega y) + \omega Q_0(y)\,\mathfrak{Sin}(\omega y) + \frac{\pi}{2}i\int_0^\infty(\omega-\varpi)H_0^{(2)}(\varpi x_0)\,\mathfrak{Cof}[(\omega-\varpi)y]\,d\varpi \quad (1,16)$$

$$\int_1^\infty \frac{e^{-\omega i x_0 s}}{x_0^2 s^2 + 1 - x_0^2} \cdot \frac{ds}{\sqrt{s^2-1}} = -i \int_0^\omega d\omega \int_1^\infty \frac{x_0 s \cdot e^{-\omega i x_0 s}}{x_0^2 s^2 + 1 - x_0^2} \cdot \frac{ds}{\sqrt{s^2-1}} + \frac{1}{y} Q_0(y) \qquad (1,17)$$

$$\int_1^\infty \frac{x_0 s \cdot e^{-\omega i x_0 s}}{(x^2 s^2 + 1 - x_0^2)^2} \cdot \frac{ds}{\sqrt{s^2-1}} = \frac{1}{2iy} \Big(-Q_1(y) \mathfrak{Sin}(\omega y)$$
$$+ \frac{\pi}{2} i P_1(y) \mathfrak{Cos}(\omega y) + Q_0(y) \omega \mathfrak{Cos}(\omega y) - \frac{\pi}{2} i P_0(y) \omega \mathfrak{Sin}(\omega y)$$
$$+ \frac{\pi}{2} i \int_0^\infty (\omega - \overline{\omega}) H_0^{(2)}(\overline{\omega} x_0) \mathfrak{Sin}[(\omega - \overline{\omega}) y] d\overline{\omega} \Big) \qquad (1,18)$$

The symmetric downwash functions of the second type read:

$$W_n = \frac{1}{V} e^{i(\nu t - \omega x)} (-1)^{n+1} \Big[\frac{i^n}{x_0} \int_0^\infty \frac{(\omega - \overline{\omega})^{n-1}}{(n-1)!} \overline{\omega} \frac{\pi}{2} i H_1^{(1)}(\overline{\omega} x_0) \mathfrak{Cos}[(\omega - \overline{\omega}) y] d\overline{\omega}$$
$$+ \frac{i^{n-1}}{(n-1)!} \omega^{n-1} \Big(\frac{\pi}{2} \frac{dP_1}{dy} \mathfrak{Cos}(\omega y) - i \frac{dQ_1}{dy} \mathfrak{Sin}(\omega y) \Big)$$
$$- \frac{i^{n-2}}{(n-2)!} \omega^{n-2} \Big(\frac{dQ_2}{dy} \mathfrak{Cos}(\omega y) + \frac{\pi}{2} i \frac{dP_2}{dy} \mathfrak{Sin}(\omega y) \Big)$$
$$- \frac{i^{n-3}}{(n-3)!} \omega^{n-3} \Big(\frac{\pi}{2} \frac{dP_3}{dy} \mathfrak{Cos}(\omega y) - i \frac{dQ_3}{dy} \mathfrak{Sin}(\omega y) \Big) + \dots \Big], \qquad (1,19)$$

the antisymmetrical

$$\overline{W}_n = \frac{1}{V} e^{i(\nu t - \omega x)} (-1)^n \Big[\frac{i^{n+1}}{x_0} \int_0^\omega \frac{(\omega - \overline{\omega})^{n-1}}{(n-1)!} \overline{\omega} \frac{\pi}{2} i H_1^{(1)}(\overline{\omega} x_0) \mathfrak{Sin}[(\omega - \overline{\omega}) y] d\overline{\omega}$$
$$+ \frac{i^n}{(n-1)!} \omega^{n-1} \Big(\frac{\pi}{2} \frac{dP_1}{dy} \mathfrak{Sin}(\omega y) - i \frac{dQ_1}{dy} \mathfrak{Cos}(\omega y) \Big)$$
$$- \frac{i^{n-1}}{(n-2)!} \omega^{n-2} \Big(\frac{dQ_2}{dy} \mathfrak{Sin}(\omega y) + \frac{\pi}{2} i \frac{dP_2}{dy} \mathfrak{Cos}(\omega y) \Big)$$
$$- \frac{i^{n-2}}{(n-3)!} \omega^{n-3} \Big(\frac{\pi}{2} \frac{dP_3}{dy} \mathfrak{Sin}(\omega y) - i \frac{dQ_3}{dy} \mathfrak{Cos}(\omega y) \Big) + \dots \Big] \qquad (1,20)$$

COMPENSATION CALCULATION

The downwash functions of the first type can be generally written in the form

$$W_\sigma = f_\sigma(x, y) e^{i\nu t} + g_\sigma(y, \omega) e^{i(\nu t - \omega x)} \qquad (2,1)$$

where $f_\sigma(x, y)$ is a polynomial in x and y, while $g_\sigma(y, \omega)$ contains transcendental functions. This second term is "compensated" with the downwash functions of the second type, which are of the form

$$W_n = h_n(y, \omega) e^{i(\nu t - \omega x)} \qquad (2,2)$$

The condition reads

$$W_\sigma + \sum_{n=0}^{\infty} K_{\sigma,n} W_n = f_\sigma(x, y) e^{i\nu t} \qquad (2,3)$$

that is,

$$-g_\sigma(y, \omega) = \sum_{n=0}^{\infty} K_{\sigma,n} h_n(y, \omega) \qquad (2,4)$$

To arrive at a numerical result for the coefficients K_σ, n, the infinite series is broken off with $n = 4$. Furthermore, equation (2.4) is multiplied by $\cos^2\alpha \cos 2k\alpha d\alpha$ $(k = 0, 1, 2)$ in the case of the symmetric downwash functions and in the antisymmetric case by $\frac{\cos^2\alpha}{\sin\alpha} \cos 2k\alpha d\alpha$ $(k = 0.1)$ and then integrated from 0 to $\frac{\pi}{2}$. This affords a first equation system for the $K_{\sigma, n}$:

$$-\int_0^{\frac{\pi}{2}} g_n(\omega, \sin\alpha)\cos^2\alpha\cos 2k\alpha\, d\alpha = \sum_n K_{\sigma,n}\int_0^{\frac{\pi}{2}} h_n(\omega, \sin\alpha)\cos^2\alpha\cos 2k\alpha\, d\alpha \qquad k = 0; 1; 2 \ldots \quad (2,5)$$

and in the antisymmetric case:

$$-\int_0^{\frac{\pi}{2}} \bar{g}_\sigma \frac{\cos^2\alpha}{\sin\alpha}\cos 2k\alpha\, d\alpha = \sum_n \bar{K}_{\sigma,n}\int_0^{\frac{\pi}{2}} \bar{h}_n(\omega, \sin\alpha)\frac{\cos^2\alpha}{\sin\alpha}\cos 2k\alpha\, d\alpha \qquad k = 0; 1 \ldots \quad (2,6)$$

THE FLOW-OFF CONDITION

Moreover, the coefficients $K_{\sigma,n}$ of the potential functions of the second type must be chosen so that the lift density at the trailing edge of the lifting surface disappears (cf. part I). The condition to be complied with is

$$K_{2\lambda+1} = \frac{4}{\pi}(-1)^{\lambda+1}(2\lambda+1)\sum_{k=0}^{\infty}\frac{(-1)^k K_{2k}}{(2\lambda+1)^2-(2k)^2}, \quad (3,1)$$

for the symmetrical case and

$$\bar{K}_{2k} = -\frac{4}{\pi}(-1)^k 2k\sum_{\lambda=0}^{\infty}\frac{(-1)^\lambda \bar{K}_{2\lambda+1}}{(2\lambda+1)^2-(2k)^2} \quad (3,2)$$

for the antisymmetrical functions.

Because only the potential functions of the second type up to $n = 4$ are considered, the infinite sums appearing in (3.1) and (3.2) should be broken off at the respective terms.

SOLUTION OF THE EQUATION SYSTEMS

If the coefficients $K_{\sigma,1}$ and $K_{\sigma,3}$ in equation (2.5) by means of (3.1) are expressed by $K_{\sigma,0}$ $K_{\sigma,2}$ and $K_{\sigma,4}$, that is,

$$\left.\begin{aligned} K_{\sigma,1} &= -\frac{4}{\pi}\left(K_{\sigma,0}+\frac{1}{3}K_{\sigma,2}-\frac{1}{15}K_{\sigma,4}\right) \\ K_{\sigma,3} &= \frac{12}{\pi}\left(\frac{1}{9}K_{\sigma,0}-\frac{1}{5}K_{\sigma,2}-\frac{1}{7}K_{\sigma,4}\right)\end{aligned}\right\} \quad (4,1)$$

while putting

$$\left.\begin{aligned} \int_0^{\frac{\pi}{2}} h_n(\omega, \sin\alpha)\cos^2\alpha\cos 2k\alpha\, d\alpha &= h_{n,k} \\ \int_0^{\frac{\pi}{2}} g_\sigma(\omega, \sin\alpha)\cos^2\alpha\cos 2k\alpha\, d\alpha &= g_{\sigma,k}\end{aligned}\right\} \quad (4,2)$$

equation (2.5) yields

$$K_{\sigma,0}\cdot M_{0,k} + K_{\sigma,2}\cdot M_{2,k} + K_{\sigma,4}\cdot M_{4,k} = -g_{\sigma,k} \qquad k = 0; 1; 2, \ldots \quad (4,3)$$

with

$$\left.\begin{aligned} M_{0,k} &= h_{0,k} - \frac{4}{\pi}h_{1,k} + \frac{4}{3\pi}h_{3,k} \\ M_{2,k} &= -\frac{4}{3\pi}h_{1,k} + h_{2,k} - \frac{12}{5\pi}h_{3,k} \\ M_{4,k} &= \frac{4}{15\pi}h_{1,k} - \frac{12}{7\pi}h_{3,k} + h_{4,k}\end{aligned}\right\} \quad (4,4)$$

The three complex equations (4.3) for the complex unknowns $K_{\sigma,0}$ $K_{\sigma,2}$, and $K_{\sigma,4}$ are divided in a real and an imaginary part, which yield the following six equations, written for brevity in matrix form:

$$\left.\begin{aligned} \mathfrak{A}\cdot\mathfrak{k}^r - \mathfrak{B}\cdot\mathfrak{k}^i &= -\mathfrak{c} \\ \mathfrak{B}\cdot\mathfrak{k}^r + \mathfrak{A}\cdot\mathfrak{k}^i &= -\mathfrak{b}\end{aligned}\right\} \quad \ldots\ldots (4,5)$$

with

$$\mathfrak{A} = \begin{pmatrix} M^r_{0,0} & M^r_{2,0} & M^r_{4,0} \\ M^r_{0,1} & M^r_{2,1} & M^r_{4,1} \\ M^r_{0,2} & M^r_{2,2} & M^r_{4,2}\end{pmatrix} \quad \mathfrak{B} = \begin{pmatrix} M^i_{0,0} & M^i_{2,0} & M^i_{4,0} \\ M^i_{0,1} & M^i_{2,1} & M^i_{4,1} \\ M^i_{0,2} & M^i_{2,2} & M^i_{4,2}\end{pmatrix} \quad (4,6)$$

$$\mathfrak{k}^r = \begin{pmatrix} K^r_{\sigma,0} \\ K^r_{\sigma,2} \\ K^r_{\sigma,4}\end{pmatrix}; \quad \mathfrak{k}^i = \begin{pmatrix} K^i_{\sigma,0} \\ K^i_{\sigma,2} \\ K^i_{\sigma,4}\end{pmatrix}; \quad \mathfrak{c} = \begin{pmatrix} g^r_{\sigma,0} \\ g^r_{\sigma,1} \\ g^r_{\sigma,2}\end{pmatrix}; \quad \mathfrak{b} = \begin{pmatrix} g^i_{\sigma,0} \\ g^i_{\sigma,1} \\ g^i_{\sigma,2}\end{pmatrix} \quad \ldots\ldots (4,7)$$

With the aid of the inverse matrices $\underline{A}^{-1}$ and $\underline{B}^{-1}$, equation (4.5) yields

$$\mathfrak{U}\cdot\mathfrak{k}^r = -(\mathfrak{B}^{-1}\mathfrak{c} + \mathfrak{A}^{-1}\mathfrak{b}) \qquad \mathfrak{U}\cdot\mathfrak{k}^i = -\mathfrak{B}^{-1}\mathfrak{b} + \mathfrak{A}^{-1}\mathfrak{c} \quad (4,8)$$

where

$$\mathfrak{U} = \mathfrak{B}^{-1}\cdot\mathfrak{A} + \mathfrak{A}^{-1}\cdot\mathfrak{B} \quad \ldots\ldots (4,9)$$

The solutions of (4.5) then follow from

$$\mathfrak{k}^r = -\mathfrak{U}^{-1}(\mathfrak{B}^{-1}\cdot\mathfrak{c} + \mathfrak{A}^{-1}\mathfrak{b}) \quad \text{und} \quad \mathfrak{k}^i = \mathfrak{U}^{-1}(\mathfrak{A}^{-1}\mathfrak{c} - \mathfrak{B}^{-1}\mathfrak{b}) \quad \ldots\ldots (4,10)$$

The calculations for the antisymmetrical motions are simpler, because the potential function $\bar{\Phi}_0$ does not exist, and so one com-

plex unknown is eliminated. Accordingly, equation (2.6) needs to be taken only for $k = 0; 1$.

Again putting

$$\left.\begin{aligned} &\int_0^{\frac{\pi}{2}} \bar{g}_\sigma(\omega, \sin\alpha)\frac{\cos^2\alpha}{\sin\alpha}\cos 2k\alpha\, d\alpha = \bar{g}_{\sigma,k};\\ &\int_0^{\frac{\pi}{2}} \bar{h}_n(\omega, \sin\alpha)\frac{\cos^2\alpha}{\sin\alpha}\cos 2k\alpha\, d\alpha = \bar{h}_{n,k}\end{aligned}\right\} \quad (4,11)$$

and

$$\left.\begin{aligned} \bar{K}_{\sigma,2} &= \frac{8}{\pi}\left(-\frac{1}{3}\bar{K}_{\sigma,1} - \frac{1}{5}\bar{K}_{\sigma,3}\right)\\ \bar{K}_{\sigma,4} &= -\frac{16}{\pi}\left(-\frac{1}{15}\bar{K}_{\sigma,1} + \frac{1}{7}\bar{K}_{\sigma,3}\right)\end{aligned}\right\} \quad (4,12)$$

according to equation (3.2), equation (2.6) becomes

$$\bar{K}_{\sigma,1}\bar{M}_{1,k} + \bar{K}_{\sigma,3}\bar{M}_{3,k} = -\bar{g}_{\sigma,k};\ k = 0;\ 1 \quad (4,13)$$

with

$$\left.\begin{aligned} \bar{M}_{1,k} &= \bar{h}_{1,k} - \frac{8}{3\pi}\bar{h}_{2,k} + \frac{16}{15\pi}\bar{h}_{4,k}\\ \bar{M}_{3,k} &= -\frac{8}{5\pi}\bar{h}_{2,k} + \bar{h}_{3,k} - \frac{16}{7\pi}\bar{h}_{4,k}\end{aligned}\right\} \quad (4,14)$$

The solution of equation (4.13) follows the same procedure as for the symmetrical case, except that all matrices are now of two rows.

The next problem is to determine the functions $g_{\sigma,k}$, $h_{n,k}$, and so forth, ((4.2), (4.11)). To this end the downwash functions enumerated earlier are developed in series of ω, the coefficients being indicated as trigonometric series in α.

The integration according to (4.2) and (4.11), yields complex series in ω for $g_{\sigma,k}$ and $h_{n,k}$, the values between $\omega = 0$ and $\omega = 2$ being computed in stages of 0.2. Then the real and the imaginary part of the equation coefficient $M_{n,k}$ can be determined with the aid of (4.4) and (4.14), respectively. Since the solution of the coefficients of the ω series $g_{\sigma,k}$ and $h_{n,k}$ involves considerable paper work, it is well to know that the calculation of the integrals for the antisymmetrical functions is rendered substantially easier if the following relations are taken into consideration (cf. Introduction):

$$\begin{aligned} &\frac{\cos^2\alpha}{\sin\alpha}\cos 2k\alpha\frac{d}{d\omega}\int_0^\omega \frac{(\omega-\varpi)^{n-1}}{(n-1)!}\varpi\frac{\pi}{2}iH_1^{(1)}(\varpi\cos\alpha)\,\mathfrak{Sin}\left[(\omega-\varpi)\sin\alpha\right]d\varpi\\ &= \frac{\cos^2\alpha}{\sin\alpha}\cos 2k\alpha\int_0^\omega \frac{(\omega-\varpi)^{n-2}}{(n-2)!}\varpi\frac{\pi}{2}iH_1^{(1)}(\varpi\cos\alpha)\,\mathfrak{Sin}\left[(\omega-\varpi)\sin\alpha\right]d\varpi\\ &+ \cos^2\alpha\cos 2k\alpha\int_0^\omega \frac{(\omega-\varpi)^{n-1}}{(n-1)!}\varpi\frac{\pi}{2}iH_1^{(1)}(\varpi\cos\alpha)\,\mathfrak{Cof}\left[(\omega-\varpi)\sin\alpha\right]d\varpi \quad (4,15)\end{aligned}$$

and

$$\frac{\cos^2\alpha}{\sin\alpha}\cos 2k\alpha\frac{d}{d\omega}\left[F(y)\,\mathfrak{Sin}(\omega\sin\alpha)\right] = \cos^2\alpha\cos 2k\alpha\, F(y)\,\mathfrak{Cof}(\omega\sin\alpha) \quad (4,16)$$

In other words, the integrals of the antisymmetrical functions can be predicted from the symmetrical functions by integrations with respect to ω. In the present report the relations (4.15) and (4.16) were used as check for the independently calculated symmetrical and antisymmetrical integrals. The results of the solution are tabulated and plotted.

DETERMINATION OF THE AIR FORCES ACTING ON THE OSCILLATING AIRFOIL

If the coefficients $K_{\sigma,n}$ derived in the foregoing section are entered in equation (2.3), it results in

$$W_\sigma + \sum_{n=0}^{4} K_{\sigma,n} W_n \sim f_\sigma(x, y)\, e^{i\nu t} \quad . \; . \; . \quad (5.1)$$

The simple periodic motions of the circular airfoil given, say, in the form of

$$z_\sigma = z_\sigma(x, y)\, e^{i\nu t} \quad . \; . \; . \; . \; . \; . \; . \quad (5.2)$$

can be computed then by linear combination of the $f_\sigma(x, y)$.

The downwash is:

$$W^{(\sigma)} = \frac{d(c z_\sigma)}{dt} = V\left(i\omega z_\sigma(x, y) + \frac{\partial z_\sigma(x, y)}{\partial x}\right) e^{i\nu t} \quad (5.3)$$

For the given flapping motion

for example, $z_A = \frac{A}{c} e^{i\nu t}$,

that is, $W_A = V\frac{A}{c} \cdot i\omega e^{i\nu t}$

and by consideration of (1.4) and (5.1) it is:

$$W_A = -\frac{2}{\pi} V^2 \cdot \frac{A}{c} i\omega \cdot f_s(x, y)\, e^{i\nu t}$$
$$\sim -\frac{2}{\pi} V^2 \cdot \frac{A}{c} i\omega \left[W_s + \sum_{n=0}^{4} K_{s,n} W_n\right]$$

Correspondingly, the respective pressure potential is

$$\varphi_A = -\frac{2}{\pi} V^2 \cdot \frac{A}{c} i\omega \left[\varphi_s + \sum_{n=0}^{4} K_{s,n}\varphi_n\right]$$
$$= V^2 \cdot \frac{A}{c} \cdot \frac{2}{\pi}\left[\omega^2 \varphi_1^0 - i\omega \sum_{n=0}^{4} K_{s,n}\varphi_n\right]$$

The pressure jump, that is, the lift density at the surface is then given by

$$\Pi(x, y, t) = 2\varrho\,\varphi_{o b},$$

which in this instance means

$$\Pi(x, y, t) = \frac{\varrho}{2} V^2 \cdot \frac{A}{c} \cdot \frac{8}{\pi}\left[\omega^2 \sqrt{1-r^2} - i\omega \sum_{n=0}^{4} K_{s,n} \frac{r^n \cos n\vartheta}{\sqrt{1-r^2}}\right] e^{i\nu t}.$$

Since only φ_1^0 and φ contribute to the total lift (cf. part I), the lift during the flapping motion is:

$$A = \pi\varrho V^2 c^2 \cdot \frac{8}{\pi}\left[\frac{1}{3}\omega^2 + \omega K_{s,0}^i - i\omega K_{s,0}^r\right] \cdot \frac{A}{c} e^{i\nu t}$$

The total moment, referred to the y-axis is given by φ_2^1 and φ_1:

$$M = \pi\varrho V^2 c^3 \cdot \frac{8}{3\pi}\left[\omega K_{s,1}^i - i\omega K_{s,1}^r\right] \cdot \frac{A}{c} e^{i\nu t}.$$

In the tables and curves the values of the dimensionless coefficients k_A and m_A are given conformably to Küssner's definition:

$$A = \pi\varrho V^2 c^2 \cdot k_A \cdot \frac{A}{c} e^{i\nu t},$$
$$M = \pi\varrho V^2 c^3 \cdot m_A \cdot \frac{A}{c} e^{i\nu t}.$$

The calculations for the relaxation oscillation are as follows:

$$z_B = -\frac{B}{c} \cdot x \cdot e^{i\nu t}.$$

whence, according to (5.3)

$$W_B = -\frac{B}{c} \cdot V \cdot (i\omega x + 1)\, e^{i\nu t}$$
$$= B \cdot V^2 i\omega \cdot \frac{4}{3\pi}\left(W_k + \sum_{n=0}^{4} K_{k,n} W_n\right)$$
$$+ B \cdot V^2 \cdot \frac{2}{\pi}\left(W_s + \sum_{n=0}^{4} K_{s,n} W_n\right).$$

With $\varphi_k = i\omega\varphi_2^1 + \frac{3}{2}\varphi_1^0$, and $\varphi_s = i\omega\varphi_1^0$

the related pressure potential reads

$$\varphi_B = \frac{B}{C} \cdot V^2 \cdot \frac{2}{\pi}\left[2 i\omega\varphi_1^0 - \frac{2}{3}\omega^2\varphi_2^1 + \sum_{n=0}^{4}\left(K_{s,n} + \frac{2}{3} i\omega K_{k,n}\right)\varphi_n\right],$$

from which pressure distribution, lift, and so forth, can be obtained again:

$$A = \pi\varrho V^2 c^2 \cdot \frac{8}{\pi}\left[-K_{s,0}^r - \frac{2}{3}\omega K_{k,0}^i + i\left(\frac{2}{3}\omega + K_{s,0}^i + \frac{2}{3}\omega K_{k,0}^r\right)\right] \cdot B \cdot e^{i\nu t}$$

$$M = \pi\varrho V^2 c^3 \cdot \frac{8}{3\pi}\left[K_{s,1}^r - \frac{2}{3}\omega K_{k,1}^i - \frac{2}{15}\omega^2 + i\left(K_{s,1}^i + \frac{2}{3}\omega K_{k,1}^r\right)\right] \cdot B \cdot e^{i\nu t}.$$

The prediction of the flexural oscillations and of the airfoil motions antisymmetrical in y follows the same procedure.

$$z_D = \frac{D}{c} \cdot x^2 \cdot e^{i\nu t} \qquad W_D = \frac{D}{c} \cdot V \cdot (i\omega x^2 + 2x)\, e^{i\nu t};$$
$$z_F = \frac{F}{c} \cdot y^2 \cdot e^{i\nu t} \qquad W_F = \frac{F}{c} \cdot V\, i\omega\, y^2 e^{i\nu t};$$
$$z_C = \frac{C}{c} \cdot y \cdot e^{i\nu t} \qquad W_C = \frac{C}{c} \cdot V\, i\omega\, y\, e^{i\nu t};$$
$$z_E = \frac{E}{c} \cdot x \cdot y \cdot e^{i\nu t} \qquad W_E = \frac{E}{c} \cdot V \cdot (i\omega x \cdot y + y)\, e^{i\nu t}.$$

The respective pressure potentials are:

$$\varphi_D = \frac{D}{c} \cdot V^2 \cdot \frac{8}{3\pi}\left[\left(-\frac{3}{2} + \frac{3}{20}\omega^2\right)\varphi_1^0 - 2i\omega\varphi_2^1 - \frac{\omega^2}{10}\varphi_3^0 + \frac{\omega^2}{5}\varphi_3^2 + \sum_{n=0}^{4}\left(-K_{K,n} + \frac{i\omega}{10}K_{B,n}^1 - \frac{i\omega}{5}K_{B,n}^2\right)\varphi_n\right],$$

$$\varphi_F = \frac{F}{c} \cdot V^2 \cdot \frac{4}{15\pi}\left[-\omega^2\varphi_3^0 + \frac{3}{2}\omega^2\varphi_1^0 - 2\omega^2\varphi_3^2 + i\omega\sum_{n=0}^{4}(K_{B,n}^1 + 2K_{B,n}^2)\,\varphi_n\right].$$

$$\varphi_C = \frac{C}{c} \cdot V^2 \cdot \frac{4}{3\pi}\left[\omega^2\bar{\varphi}_2^1 - i\omega\sum_{n=1}^{4}\bar{K}_{R,n}\bar{\varphi}_n\right].$$

$$\varphi_E = \frac{E}{c} \cdot V^2 \cdot \frac{4}{3\pi}\left[-2i\omega\bar{\varphi}_2^1 + \frac{2}{5}\omega^2\bar{\varphi}_3^2 - \sum_{n=1}^{4}\left(\bar{K}_{R,n} + \frac{2}{5}i\omega\bar{K}_{B,n}^3\right)\bar{\varphi}_n\right].$$

The potential functions φ_n^m and φ_n are given in part I. Their values at the circular surface are obtained for $\eta = 0$. The corresponding lift and moment coefficients are indicated with k_D, m_D, k_F, m_F, and so forth.

TABLES AND DIAGRAMS

Tables 1 to 6 give the values of coefficients $K_{\sigma,n}$ divided in real and imaginary part, resulting from the compensation calculation in connection with the flow-off condition. The imaginary part $K_{\sigma,n}^i$ is zero for $\omega = 0$.

The framed-in values are equal to $\lim\limits_{\omega \to 0}\left(\frac{K_{\sigma,n}^i}{\omega}\right) = \left[\frac{dK_{\sigma,n}^i}{d\omega}\right]_{\omega=0}$.

The real and imaginary part of the lift and moment coefficients are also given for the different oscillation modes; the graphical representations show these in complex representation in the Gaussian plane of numbers.

Figures 11 to 20 show in perspective representation the real and imaginary part of the pressure at $\omega = 0, 1.0, 1.8$ referred to dynamic pressure $\frac{\rho}{2}V^2$ and for the amplitudes $\frac{A}{c}$ and $\frac{B}{c} = 1$, and so forth, respectively. The values are plotted for the wing sections $\left|\frac{y}{c}\right| = 0; = \sin\frac{\pi}{8} = 0.383; = \sin\frac{\pi}{4} = 0.707;$ $\sin\frac{3\pi}{8} = 0.924$ of the right and left half of the circular airfoil. Incidentally, it should be noted that the pressure distribution at the trailing edge of the airfoil was extrapolated with respect to zero, which does not follow mathematically because of the finite number of the series terms (cf. Kinner, reference 1).

Figures 21 to 24 give the comparison of the presented calculations with the values from the strip theory and from Possio's theory (reference 6). By the strip theory each wing section is computed according to the two-dimensional theory without regard to the mutual interference of the sections and in particular to the finite span. According to it the stationary value is $\frac{dc_a}{d\alpha} = 2\pi \approx 6.28$; according to Possio, who obtains the formulas of the lifting line for $\omega = 0$, it is $\frac{dc_a}{d\alpha} = \frac{2\pi\lambda}{\lambda+2} \approx 2.44$. The premises for his calculation were: great aspect ratio λ and small ω values. According to circular airfoil theory (reference 1) the stationary value is $\frac{dc_a}{d\alpha} = 1.82$.

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Table I. Flapping oscillation: $z_A = \frac{A}{c} e^{i\nu t}$.

Coefficients $K_{s,n} = K^r_{s,n} + i\, K^i_{s,n}$.

ω	0	0,2	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
$K^r_{s,0}$	0,3531	0,3456	0,3251	0,2905	0,2413	0,1786	0,1056	0,0276	−0,0484	−0,1139	−0,1576
$K^r_{s,1}$	−0,5489	−0,5357	−0,4988	−0,4367	−0,3487	−0,2373	−0,1086	0,0281	0,1613	0,2789	0,3685
$K^r_{s,2}$	0,2246	0,2155	0,1891	0,1450	0,0837	0,0086	−0,0753	−0,1609	−0,2412	−0,3116	−0,3751
$K^r_{s,3}$	0,0042	0,0090	0,0234	0,0466	0,0766	0,1098	0,1416	0,1668	0,1806	0,1797	0,1640
$K^r_{s,4}$	−0,0475	−0,0493	−0,0547	−0,0624	−0,0699	−0,0743	−0,0720	−0,0590	−0,0308	0,0183	0,1020
	s. S. 286										
$K^i_{s,0}$	−0,2484	−0,0522	−0,1083	−0,1645	−0,2154	−0,2553	−0,2787	−0,2813	−0,2604	−0,2151	−0,1462
$K^i_{s,1}$	0,4465	0,0931	0,1913	0,2879	0,3737	0,4387	0,4739	0,4725	0,4314	0,3506	0,2330
$K^i_{s,2}$	−0,3052	−0,0622	−0,1243	−0,1813	−0,2271	−0,2555	−0,2617	−0,2432	−0,2005	−0,1370	−0,0582
$K^i_{s,3}$	0,1235	0,0241	0,0448	0,0587	0,0626	0,0539	0,0311	−0,0052	−0,0524	−0,1059	−0,1599
$K^i_{s,4}$	0,0078	0,0023	0,0078	0,0183	0,0356	0,0604	0,0927	0,1312	0,1740	0,2185	0,2607

Lift and moment coefficient ($k_A = k'_A + i\, k''_A$, $m_A = m'_A + i\, m''_A$.)

k'_A	0	0,0073	0,0255	0,0543	0,1044	0,1986	0,3705	0,6608	1,1120	1,7641	2,6507
k''_A	0	−0,1760	−0,3311	−0,4439	−0,4916	−0,4547	−0,3226	−0,0984	0,1970	0,5222	0,8025
m'_A	0	0,0158	0,0650	0,1466	0,2538	0,3724	0,4827	0,5615	0,5859	0,5357	0,3956
m''_A	0	0,0909	0,1694	0,2224	0,2368	0,2014	0,1106	−0,0334	−0,2191	−0,4261	−0,6255

Table II. Relaxation oscillation: $z_B = \frac{B}{c} \cdot x \cdot e^{i\nu t}$.

Coefficients $K_{K,n} = K^r_{K,n} + i\, K^i_{K,n}$.

ω	0	0,2	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
$K^r_{K,0}$	−0,2221	−0,2212	−0,2133	−0,1941	−0,1613	−0,1153	−0,0583	0,0049	0,0681	0,1247	0,1651
$K^r_{K,1}$	0,3872	0,3844	0,3677	0,3299	0,2681	0,1831	0,0797	−0,0336	−0,1462	−0,2470	−0,3258
$K^r_{K,2}$	−0,2446	−0,2397	−0,2213	−0,1861	−0,1338	−0,0668	0,0100	0,0895	0,1646	0,2282	0,2824
$K^r_{K,3}$	0,0888	0,0829	0,0648	0,0351	−0,0041	−0,0490	−0,0945	−0,1345	−0,1636	−0,1767	−0,1729
$K^r_{K,4}$	0,0069	0,0117	0,0252	0,0453	0,0694	0,0937	0,1138	0,1251	0,1223	0,1013	0,0499
	s. S. 286										
$K^i_{K,0}$	0,1259	0,0302	0,0686	0,1113	0,1526	0,1865	0,2074	0,2107	0,1934	0,1543	0,0927
$K^i_{K,1}$	−0,2630	−0,0603	−0,1330	−0,2109	−0,2844	−0,3433	−0,3780	−0,3814	−0,3499	−0,2839	−0,1857
$K^i_{K,2}$	0,2597	0,0549	0,1132	0,1698	0,2181	0,2515	0,2648	0,2550	0,2224	0,1724	0,1124
$K^i_{K,3}$	−0,1934	−0,0382	−0,0730	−0,1006	−0,1171	−0,1191	−0,1045	−0,0734	−0,0283	0,0256	0,0819
$K^i_{K,4}$	0,0887	0,0165	0,0286	0,0332	0,0280	0,0111	−0,0179	−0,0585	−0,1092	−0,1683	−0,2354

Lift and moment coefficients

k'_B	0,8992	0,8699	0,7813	0,6264	0,4073	0,1381	−0,1537	−0,4304	−0,6484	−0,7617	−0,7160
k''_B	0	0,1314	0,2584	0,4020	0,5904	0,8517	1,2085	1,6720	2,2380	2,8889	3,5835
m'_B	−0,4659	−0,4524	−0,4114	−0,3398	−0,2397	−0,1203	0,0015	0,1043	0,1640	0,1592	0,0703
m''_B	0	0,1225	0,2456	0,3564	0,4386	0,4760	0,4563	0,3745	0,2339	0,0460	−0,1710

Table III. Flexural oscillation 2: $W_B^2 = F_1 (x^2 - y^2) e^{i \nu t}$.

Coefficients $K_{B,n}^2 = K_{B,n}^{2(r)} + i K_{B,n}^{2(i)}$.

ω	0	0,2	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
$K_{B,0}^{2(r)}$	0,1123	0,1077	0,0937	0,0701	0,0372	—0,0034	—0,0487	—0,0949	—0,1378	—0,1757	—0,2090
$K_{B,1}^{2(r)}$	—0,2620	—0,2531	—0,2266	—0,1823	—0,1212	—0,0471	0,0338	0,1133	0,1825	0,2349	0,2655
$K_{B,2}^{2(r)}$	0,3034	0,2944	0,2683	0,2257	0,1685	0,1015	0,0311	—0,0341	—0,0850	—0,1121	—0,1035
$K_{B,3}^{2(r)}$	—0,2465	—0,2369	—0,2075	—0,1602	—0,0982	—0,0256	0,0516	0,1269	0,1936	0,2456	0,2764
$K_{B,4}^{2(r)}$	0,1144	0,1057	0,0776	0,0322	—0,0271	—0,0978	—0,1759	—0,2586	—0,3429	—0,4298	—0,5243
	s. S. 286										
$K_{B,0}^{2(i)}$	—0,1580	—0,0323	—0,0646	—0,0945	—0,1185	—0,1334	—0,1365	—0,1263	—0,1090	—0,0678	—0,0217
$K_{B,1}^{2(i)}$	0,3199	0,0648	0,1286	0,1863	0,2313	0,2576	0,2602	0,2369	0,1884	0,1202	0,0402
$K_{B,2}^{2(i)}$	—0,3360	—0,0670	—0,1305	—0,1851	—0,2253	—0,2456	—0,2425	—0,2147	—0,1634	—0,0979	—0,0323
$K_{B,3}^{2(i)}$	0,3433	0,0678	0,1304	0,1826	0,2197	0,2374	0,2334	0,2069	0,1592	0,0956	0,0229
$K_{B,4}^{2(i)}$	—0,2817	—0,0556	—0,1066	—0,1489	—0,1794	—0,1950	—0,1944	—0,1768	—0,1430	—0,0909	—0,0137

Bending oscillation 1: $W_B^1 = D_1 (x^2 + y^2) e^{i \nu t}$.

For the coefficients $K_{B,n}^1$ the relation (c.f. (1.8)): $K_{B,n}^1 = -\frac{5}{2} K_{s,n}$ is applicable.

Table IV. Flexural oscillation: $z_D = \frac{D}{c} \cdot x^2 \cdot e^{i \nu t}$.

Lift and moment coefficients

ω	0	0,2	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
k_D'	—0,9436	—0,9531	—1,0005	—1,0999	—1,2519	—1,4439	—1,6503	—1,8359	—1,9598	—1,9824	—1,8568
k_D''	0	0,0291	0,0970	0,2012	0,3340	0,4840	0,6363	0,7727	0,8720	0,9129	0,8661
m_D	—0,4382	—0,4268	—0,3828	—0,2992	—0,1770	—0,0248	0,1414	0,3002	0,4289	0,5071	0,5188
m_D''	0	0,0202	0,0464	0,0660	0,0606	0,0136	—0,0878	—0,2491	—0,4675	—0,7312	—1,0239

Bending oscillation: $z_F = \frac{F}{c} \cdot y^2 \cdot e^{i \nu t}$.

Lift and moment coefficients

ω	0	0,2	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
k_F'	0	0,0023	0,0079	0,0158	0,0267	0,0436	0,0718	0,1185	0,1928	0,3042	0,4603
k_F''	0	—0,0441	—0,0849	—0,1194	—0,1437	—0,1539	—0,1472	—0,1230	—0,0841	—0,0407	—0,0163
m_F'	0	0,0023	0,0100	0,0236	0,0427	0,0658	0,0902	0,1121	0,1271	0,1296	0,1137
m_F''	0	0,0189	0,0359	0,0494	0,0570	0,0565	0,0461	0,0248	—0,0070	—0,0463	—0,0883

Table V. Roll oscillation: $z_C = \frac{C}{c} \cdot y \cdot e^{i\nu t}$.
Coefficients $\overline{K}_{R,n} = \overline{K}^r_{R,n} + i\,\overline{K}^i_{R,n}$.

ω	0	0,2	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
$K^r_{R,1}$	0,2254	0,2177	0,1965	0,1625	0,1197	0,0734	0,0300	−0,0045	−0,0256	−0,0324	−0,0254
$K^r_{R,2}$	−0,2448	−0,2351	−0,2074	−0,1638	−0,1090	−0,0497	0,0067	0,0527	0,0828	0,0940	0,0889
$K^r_{R,3}$	0,1050	0,0987	0,0798	0,0508	0,0146	−0,0249	−0,0631	−0,0960	−0,1201	−0,1325	−0,1324
$K^r_{R,4}$	0,0001	0,0021	0,0086	0,0182	0,0300	0,0430	0,0561	0,0683	0,0787	0,0854	0,0877
$K^i_{R,1}$	s. S. 286 −0,1876	−0,0368	−0,0702	−0,0962	−0,1109	−0,1118	−0,0981	−0,0721	−0,0390	−0,0029	0,0294
$K^i_{R,2}$	0,2593	0,0509	0,0968	0,1322	0,1525	0,1545	0,1374	0,1039	0,0594	0,0094	−0,0383
$K^i_{R,3}$	−0,1965	−0,0387	−0,0730	−0,0993	−0,1146	−0,1170	−0,1063	−0,0837	−0,0516	−0,0136	0,0263
$K^i_{R,4}$	0,0793	0,0157	0,0292	0,0396	0,0457	0,0471	0,0440	0,0364	0,0243	0,0089	−0,0091

Rolling moment coefficient $m_C = m'_C + i\,m''_C$.

m'_C	0	−0,0004	−0,0022	−0,0081	−0,0222	−0,0499	−0,0963	−0,1647	−0,2544	−0,3638	−0,4859
m''	0	0,0246	0,0445	0,0552	0,0542	0,0416	0,0204	−0,0036	−0,0231	−0,0330	−0,0287

Table VI. Torsional oscillation: $z_E = \frac{E}{c} \cdot x \cdot y \cdot e^{i\nu t}$.
Coefficients $\overline{K}^3_{B,n} = \overline{K}^{3(r)}_{B,n} + i\,\overline{K}^{3(i)}_{B,n}$.

ω	0	0,2	0,4	0,6	0,8	1,0	1,2	1,4	1,6	1,8	2,0
$K^{3(r)}_{B,1}$	−0,2431	−0,2328	−0,2059	−0,1623	−0,1068	−0,0467	0,0093	0,0526	0,0758	0,0766	0,0553
$K^{3(r)}_{B,2}$	0,3301	0,3148	0,2721	0,2045	0,1190	0,0258	−0,0629	−0,1356	−0,1821	−0,1973	−0,1805
$K^{3(r)}_{B,3}$	−0,2430	−0,2301	−0,1911	−0,1310	−0,0556	0,0270	0,1080	0,1785	0,2312	0,2598	0,2622
$K^{3(r)}_{B,4}$	0,0943	0,0884	0,0691	0,0402	0,0042	−0,0355	−0,0754	−0,1120	−0,1425	−0,1631	−0,1720
$K^{3(i)}_{B,1}$	s. S. 286 0,2344	0,0458	0,0873	0,1187	0,1346	0,1314	0,1063	0,0650	0,0120	−0,0456	−0,0966
$K^{3(i)}_{B,2}$	−0,4101	−0,0804	−0,1528	−0,2084	−0,2396	−0,2410	−0,2101	−0,1526	−0,0747	0,0132	0,0979
$K^{3(i)}_{B,3}$	0,4145	0,0817	0,1545	0,2114	0,2460	0,2542	0,2354	0,1913	0,1267	0,0500	−0,0313
$K^{3(i)}_{B,4}$	−0,2220	−0,0439	−0,0828	−0,1135	−0,1333	−0,1403	−0,1352	−0,1171	−0,0881	−0,0519	−0,0100

Rolling moment coefficient $m_E = m_E + i\,m''_E$.

m'_E	0,1276	0,1211	0,1033	0,0759	0,0434	0,0118	−0,0119	−0,0232	−0,0188	0,0002	0,0294
m''_E	0	0,0139	0,0322	0,0593	0,0990	0,1525	0,2186	0,2927	0,3676	0,4370	0,4944

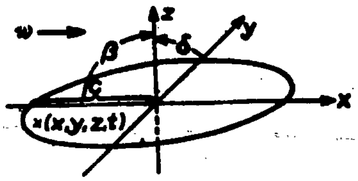

Figure 1.- β and δ indicate the positive direction of the moments.

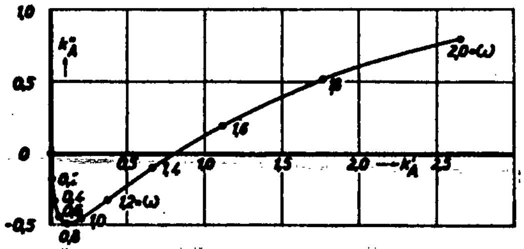

Figure 2.- Flapping oscillation A; lift coefficients in complex representation.

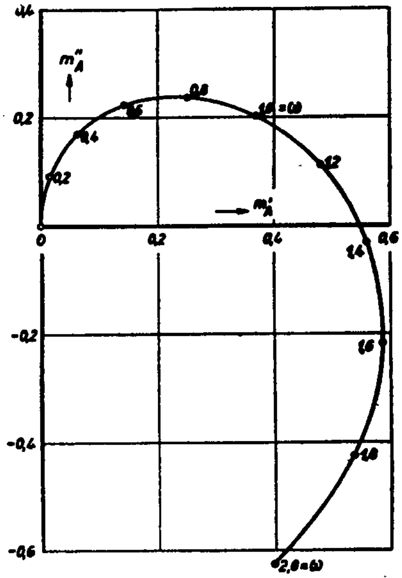

Figure 3.- Flapping oscillation A; moment coefficients in complex representation.

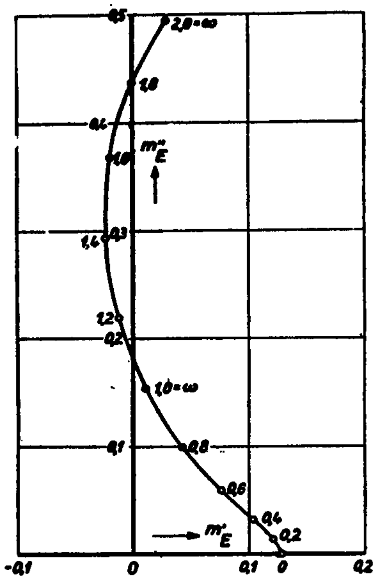

Figure 4.- Relaxation oscillation B·x, lift coefficients in complex representation.

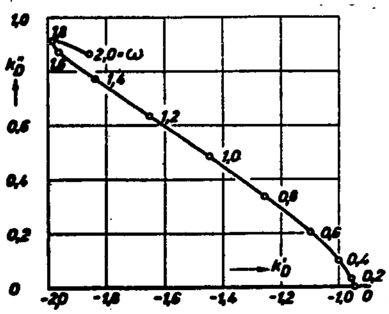

Figure 6.- Bending oscillation D·x^2; lift coefficients in complex representation.

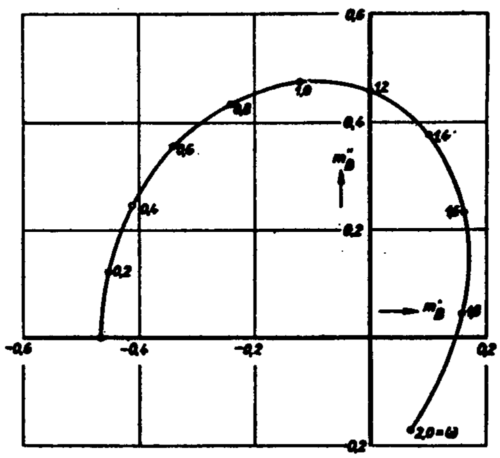

Figure 5.- Relaxation oscillation in B·x, moment coefficients in complex representation.

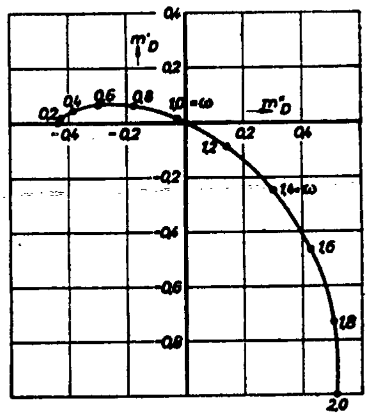

Figure 7.- Bending oscillation $D \cdot x^2$; moment coefficients in complex representation.

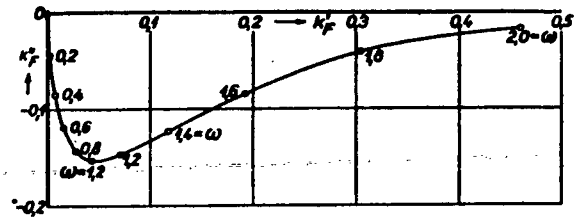

Figure 8.- Bending oscillation $F \cdot y^2$; lift coefficients in complex representation.

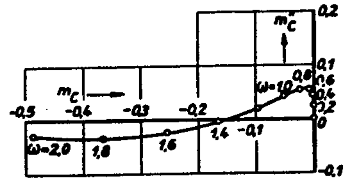

Figure 10.- Rolling oscillation $C \cdot y$; rolling moment in complex representation.

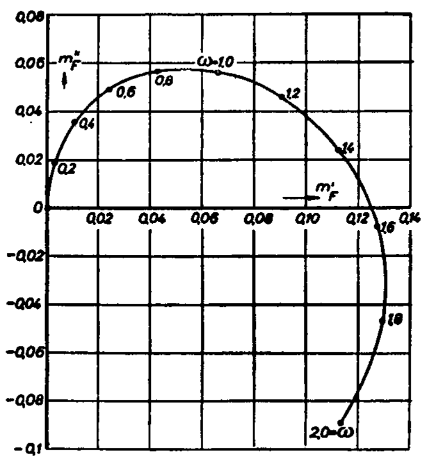

Figure 9.- Bending oscillation $F \cdot y^2$; moment coefficients in complex representation.

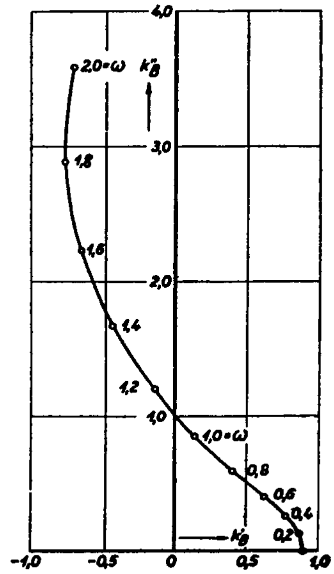

Figure 11.- Torsional oscillation $E \cdot x \cdot y$; rolling moment in complex representation.

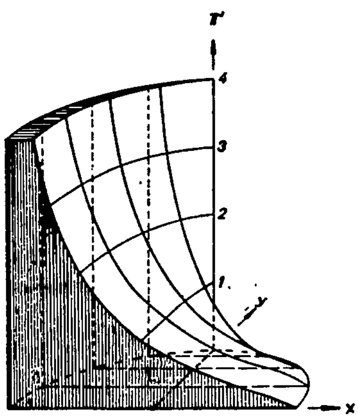

Figure 12.- Flapping oscillation A; limit case of imaginary pressure distribution.

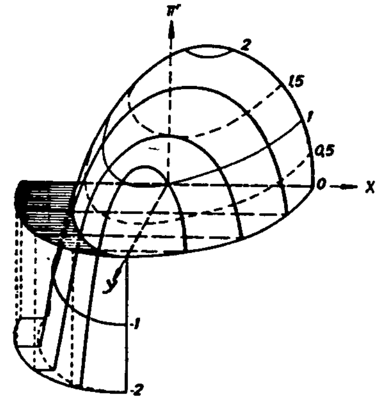

Figure 13.- Flapping oscillation A; real pressure distribution at $\omega = 1$.

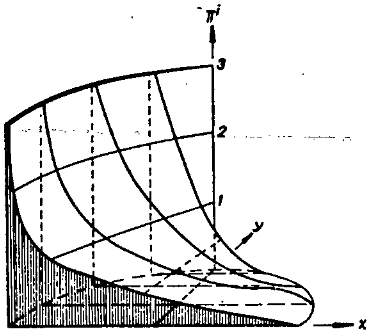

Figure 14.- Flapping oscillation A; imaginary pressure distribution at ω = 1.

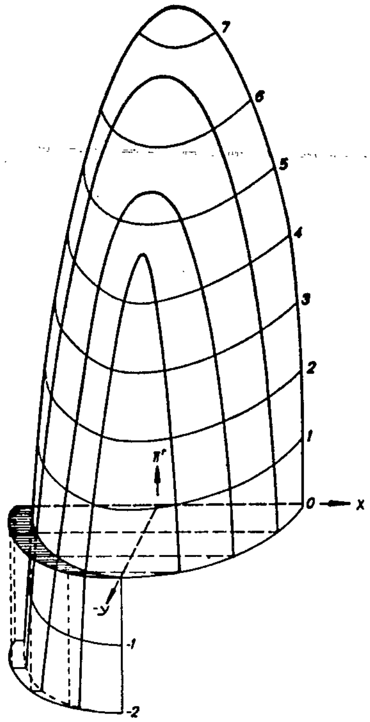

Figure 15.- Flapping oscillation A; real pressure distribution at ω = 1.8.

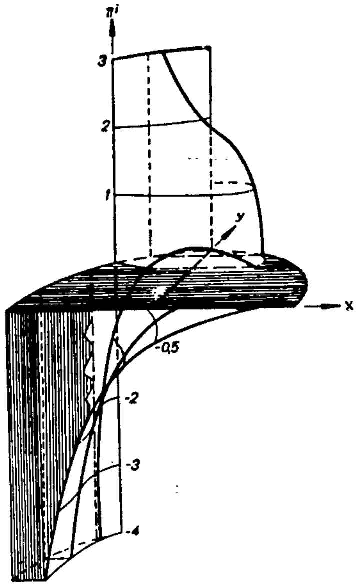

Figure 16.- Flapping oscillation A; imaginary pressure distribution at ω = 1.8.

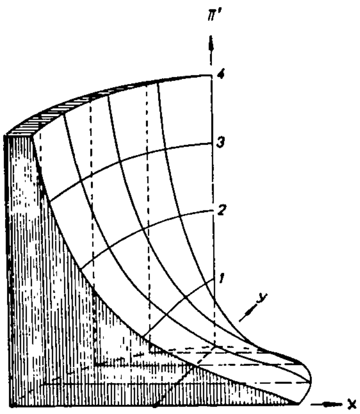

Figure 17.- Relaxation oscillation B·x; real pressure distribution at ω = 0.

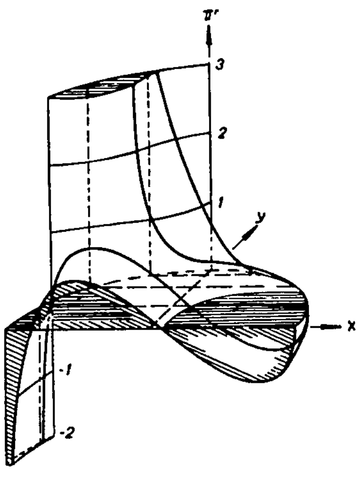

Figure 18.- Relaxation oscillation B·x; real pressure distribution at ω = 1.0.

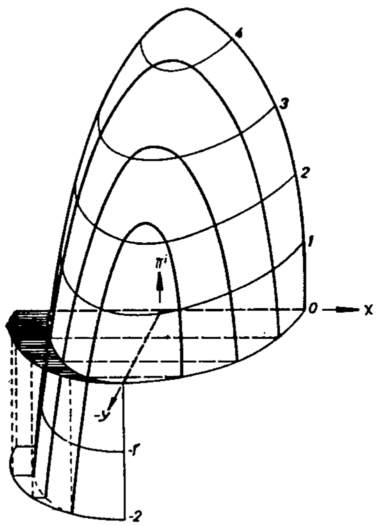

Figure 19.- Relaxation oscillation B·x; imaginary pressure distribution at ω = 1.0.

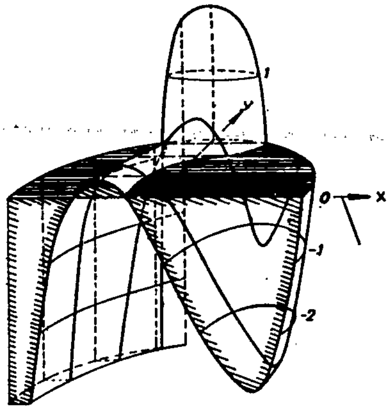

Figure 20.- Relaxation oscillation B·x; real pressure distribution at ω = 1.8.

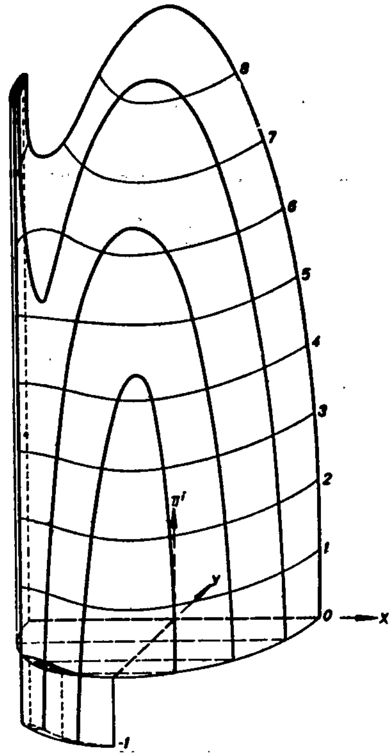

Figure 21.- Relaxation oscillation B·x; imaginary pressure distribution at ω = 1.8.

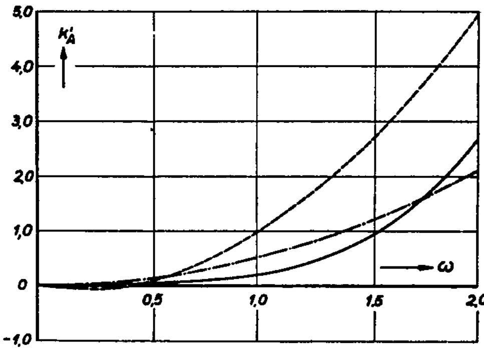

Figure 22.- Flapping oscillation A; lift coefficient k_A' (real part) plotted against
——— present calculation,
—·—·— Possio's theory,
------- strip theory (two-dimensional flow).

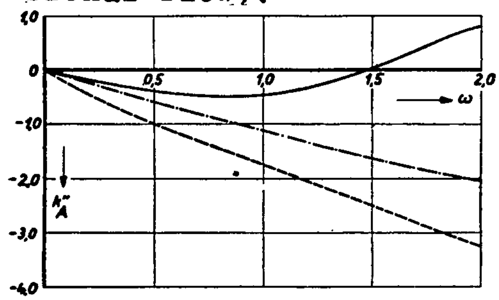

Figure 23.- Flapping oscillation A; lift coefficient k_A'' (imaginary part). (cf fig. 21.)

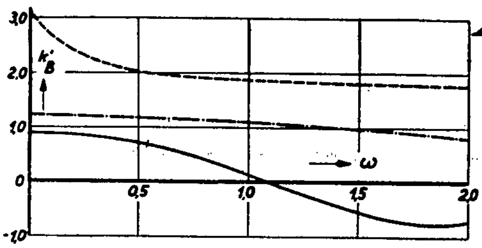

Figure 24.- Relaxation oscillation B·x; lift coefficient k_B' (real part).(of fig. 21.)

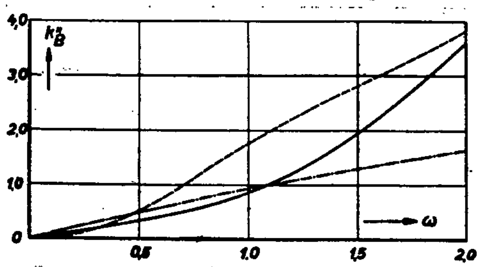

Figure 25.- Relaxation oscillation B·x; lift coefficient k_B'' (imaginary part).